Part IX

第九编

Loop Quantum Gravity and Spinfoams

圈量子引力与自旋泡沫

Abhay Ashtekar and Yongge Ma

阿贝·阿西提卡、马永革

83 Emergence of Riemannian Quantum Geometry

83 黎曼量子几何的涌现

Hal M. Haggard, Jerzy Lewandowski, and Hanno Sahlmann

哈尔·M. 哈格德，耶日·莱万多夫斯基，汉诺·萨尔曼

Contents

目录

H. M. Haggard

哈尔·M. 哈格德

Bard College, Annandale-On-Hudson, NY, USA

美国纽约州安嫩代尔-昂德-哈德逊巴德学院

e-mail: hhaggard@bard.edu

电子邮箱:hhaggard@bard.edu

J. Lewandowski (☑)

耶日·莱万多夫斯基 (☑)

Faculty of Physics, University of Warsaw, Warsaw, Poland

波兰华沙大学物理系

e-mail: jerzy.lewandowski@fuw.edu.pl

电子邮箱:jerzy.lewandowski@fuw.edu.pl

H. Sahlmann

汉诺·萨尔曼

Department Physik, Friedrich-Alexander Universität Erlangen-Nürnberg (FAU), Erlangen, Germany

德国埃尔兰根弗里德里希-亚历山大埃尔朗根-纽伦堡大学 (FAU) 物理系

e-mail: hanno.sahlmann@fau.de; hanno.sahlmann@gravity.fau.de

电子邮箱:hanno.sahlmann@fau.de; hanno.sahlmann@gravity.fau.de

Abstract

摘要

In this chapter, we take up the quantum Riemannian geometry of a spatial slice of spacetime. While researchers are still facing the challenge of observing quantum gravity, there is a geometrical core to loop quantum gravity that does much to define the approach. This core is the quantum character of its geometrical observables: space and spacetime are built up out of Planck-scale quantum grains. The interrelations between these grains are described by spin networks, graphs whose edges capture the bounding areas of the interconnected nodes, which encode the extent of each grain. We explain how quantum Riemannian geometry emerges from two different approaches: in the first half of the chapter, we take the perspective of continuum geometry and explain how quantum geometry emerges from a few principles, such as the general rules of canonical quantization of field theories, a classical formulation of general relativity in which it appears embedded in the phase space of Yang-Mills theory, and general covariance. In the second half of the chapter, we show that quantum geometry also emerges from the direct quantization of the finite number of degrees of freedom of the gravitational field encoded in discrete geometries. These two approaches are complementary and are offered to assist readers with different backgrounds enter the compelling arena of quantum Riemannian geometry.

在本章中，我们将探讨时空空间切片的量子黎曼几何。尽管研究者仍面临观测量子引力的挑战，但圈量子引力存在一个几何核心，该核心在很大程度上定义了这一研究方向。这个核心就是其几何可观测量的量子特性: 空间与时空由普朗克尺度的量子晶粒构成。这些晶粒之间的相互关系由自旋网络描述，自旋网络是一类图，其边刻画了相互连接的节点的包围面积，节点则编码了每个晶粒的体积范围。我们将阐释量子黎曼几何如何从两种不同方法中涌现出来: 在本章前半部分，我们从连续几何的视角出发，解释量子几何如何从若干原理中涌现，这些原理包括场论正则量子化的一般规则、广义相对论嵌入杨-米尔斯理论相空间的经典表述，以及广义协变性。在本章后半部分，我们表明量子几何同样可以从离散几何所编码的引力场有限自由度直接量子化中涌现。这两种方法互为补充，旨在帮助不同研究背景的读者进入引人入胜的量子黎曼几何领域。

Keywords

关键词

Quantum gravity · Quantum geometry · Loop quantum gravity · Canonical quantum gravity · Discrete quantum geometry · Geometrical operators

量子引力 · 量子几何 · 圈量子引力 · 正则量子引力 · 离散量子几何 · 几何算符

Introduction

引言

Since general relativity is a theory of the geometry of space and time as much as it is a theory of gravity, it seems evident that a quantum theory of geometry of some kind will be one aspect of a theory of quantum gravity. The present chapter is about the (spatial part of) the quantum geometry that arises in the quantization of gravity pursued in loop quantum gravity.

由于广义相对论既是引力理论，也是时空几何理论，因此显然，某种量子几何理论必然是量子引力理论的一个组成部分。本章介绍圈量子引力研究框架下，引力量子化过程中涌现出的量子几何 (空间部分)。

The first half of the chapter focuses on a continuum approach and arises from a few principles with surprisingly little room for modifications. These principles are the general rules of canonical quantization of field theories, a classical formulation of general relativity in which it appears embedded in the phase space of Yang-Mills theory, and the principle of general covariance. We will explain in detail in this chapter how quantum Riemannian geometry arises from this continuum approach to loop quantum gravity.

本章前半部分聚焦连续统方法，该方法基于少数几条基本原理，几乎没有修改空间。这些原理分别是: 场论正则量子化的一般规则、嵌入杨-米尔斯理论相空间的广义相对论经典表述，以及广义协变原理。我们将在本章中详细说明量子黎曼几何如何从圈量子引力的这种连续统方法中涌现。

Remarkably, quantum Riemannian geometry arises also in other contexts: many aspects of it were anticipated already by Penrose decades before the advent of loop quantum gravity, and a direct approach that quantizes discrete classical geometries is both illuminating and surprisingly rich. In particular, this approach looks to the finite number of degrees of freedom of the gravitational field that are captured by the geometry of Euclidean polyhedra. We also explain how the quantization of these polyhedra gives another road to the emergence of quantum Riemannian geometry in the second half of this chapter.

值得注意的是，量子黎曼几何也会在其他研究背景中出现: 它的许多核心内容早在圈量子引力诞生数十年前就被彭罗斯预见，而对离散经典几何进行直接量子化的研究方法不仅富有启发，内容也意外丰富。具体而言，该方法关注欧几里得多面体几何所捕获的引力场有限自由度。我们还会在本章后半部分说明，这些多面体的量子化如何为量子黎曼几何的涌现开辟另一条路径。

The states of quantum geometry consist of seemingly one-dimensional excitations. A basis is given by spin network states, described by a graph, decorated with irreducible representations of SU (2) on the edges, and invariant tensors on the vertices. Remarkably they can also be read as linear combinations of quantum circuits or as (again a linear combination of) a sort of Feynman diagram. In spin networks, the group SU(2) takes the role that Poincaré symmetry takes in Feynman diagrams, and the interaction vertices describe the formation of spatial volume, not the scattering and decay of particles.

量子几何的态由表观上一维的激发构成。其一组基由自旋网络态给出，自旋网络态由图描述，边带有 SU (2) 的不可约表示标记，顶点带有不变张量标记。值得注意的是，它们也可被解读为量子线路的线性组合，或是某种费曼图的线性组合。在自旋网络中，SU(2) 群的作用对应庞加莱对称性在费曼图中的作用，相互作用顶点描述空间体积的形成，而非粒子的散射和衰变。

One basic aspect of this geometry is the discreteness of the spectra of geometric operators, in particular that of area. A spin network edge decorated with the spin- j representation contributes a quantum of area

这种几何的一个基本特征是几何算符谱的离散性，面积算符的谱尤为明显。带有自旋- j 表示标记的自旋网络边，会贡献一个面积量子

$$a_j = 8\pi\gamma\ell_P^2\sqrt{j\,(j+1)} \tag{1}$$

to any surface traversed by the edge. Here

给该边穿过的任意曲面。这里

$$\ell_P^2 = \frac{\hbar G}{c^3} \approx 2.6 \times 10^{-70}\ \mathrm{m}^2 \tag{2}$$

is the Planck area, and γ is a parameter of the theory. This stunning scale explains the difficulty of observing quantum geometry and sets the stage for the challenge of finding its observable consequences.

是普朗克面积，γ 是该理论的一个参数。这个极小的尺度解释了观测量子几何为何如此困难，也为寻找其可观测效应的研究设立了挑战。

We lay out some details of the emergence of quantum Riemannian geometry in loop quantum gravity sections on ”The Classical Holonomy-Flux Variables” and ”Ashtekar-Lewandowski Representation” and its properties in the section ”Quantum Geometry”. We discuss the emergence of quantum Riemannian geometry from quantizing discrete geometry in the section on ”Discrete Geometry”. The literature addressing both halves of this chapter is vast; rather than attempting a fully rigorous and complete account, we have opted to try to make this chapter more accessible to a researcher new to the field. We encourage readers to explore the multitude of references provided throughout the chapter for further details.

我们在“经典全纯-通量变量”小节阐述了圈量子引力中量子黎曼几何涌现的部分细节，在“阿西卡-莱万多夫斯基表示”小节介绍了相关内容，在“量子几何”小节讨论了它的性质。我们在“离散几何”小节讨论了离散几何量子化中量子黎曼几何的涌现。涵盖本章两部分内容的相关文献数量庞大；我们没有追求全面严谨的完整论述，而是选择让本章节更便于刚接触该领域的研究者阅读。我们鼓励读者阅读本章各处提供的大量参考文献，了解更多细节。

The Holonomy-Flux Variables for General Relativity

广义相对论的全纯-通量变量

In this first section, we present an account of the quantization of general relativity, that underlying loop quantum gravity. The result is, among other things, a quantum theory of intrinsic and extrinsic Riemannian geometry.

在第一节中，我们介绍圈量子引力基础的广义相对论量子化方法。该成果最终得到了一个关于内蕴与外蕴黎曼几何的量子理论。

The formalism is based on a phase space formulation that embeds general relativity into the phase space of SU(2) Yang-Mills theory and an operator algebra and Hilbert space that uses no auxiliary classical structures such as a flat background metric. Therefore, all the structures transform covariantly under the action of diffeomorphisms.

该形式体系基于将广义相对论嵌入 SU(2) 杨-米尔斯理论相空间的相空间公式，其算子代数与希尔伯特空间不使用平坦背景度规这类辅助经典结构，因此所有结构在微分同胚作用下都满足协变变换。

We will first describe the classical setup in sections on ”General Relativity Inside the Phase Space of Yang-Mills Theory” and ”The Classical Holonomy-Flux Variables”. Then we will come to quantization (section ”The Quantum Holonomy-Flux Variables”) and finally to the resulting quantum geometry in sections ”Ashtekar-Le-wandowski Representation” and ”Quantum Geometry”. More extensive accounts of the theory covered in the following sections can be found in [1-5].

我们首先在「杨-米尔斯理论相空间内的广义相对论」和「经典全纯-通量变量」小节中描述经典构型，之后进入量子化环节（「量子全纯-通量变量」小节），最后在「阿西卡-莱万多夫斯基表示」和「量子几何」小节中介绍由此得到的量子几何。后续章节涵盖的理论更详细的阐述可参见文献 [1-5]。

General Relativity Inside the Phase Space of Yang-Mills Theory

杨-米尔斯理论相空间内的广义相对论

Consider a fixed 4-manifold $\mathcal{M}$. Let $e^I = e^0, \dots, e^3$ be coframes, and construct the metric tensor

考虑一个固定的 4 维流形 $\mathcal{M}$ 。设 $e^I = e^0, \dots, e^3$ 为余标架，构造度量张量

$$g = \eta_{IJ} e^I e^J \tag{3}$$

on $\mathcal{M}$; here $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$, and is used to raise and lower capital Latin indices. An SO(3,1) connection ω , upon choice of a gauge, can be written as a matrix of 1-forms ${\omega^I}_J, I, J = 0, \dots, 3$, such that $\omega_{IJ} = -\omega_{JI}$. The curvature 2-form of ${\omega^I}_J$ is

于 $\mathcal{M}$ 上；此处为 $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$ ，用于升降大写拉丁指标。一个 SO(3,1) 联络 ω 在选定规范后，可以写为 1-形式矩阵 ${\omega^I}_J, I, J = 0, \dots, 3$ ，满足 $\omega_{IJ} = -\omega_{JI}$ 。${\omega^I}_J$ 的曲率 2 形式为

$${\Omega^I}_J \ := \ d{\omega^I}_J + {\omega^I}_K \wedge {\omega^K}_J.$$

With these definitions, the Palatini-Holst action [6] is defined by

根据上述定义，帕拉蒂尼-霍尔斯特作用量 [6] 定义为

$$S_{\mathrm{PH}}(e,\omega) = \frac{1}{4\kappa}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} - \frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{KL}, \tag{4}$$

where $\kappa := 8\pi G$ in units with $c = 1$ and $\gamma > 0$ is the Barbero-Immirzi parameter, the significance of which will emerge below. The covariant symplectic form [7] derived from S_{PH} is

其中 $\kappa := 8\pi G$ 在取 $c = 1$ 和 $\gamma > 0$ 为单位的单位制中是巴贝罗-伊米尔齐参数，该参数的重要性我们将在后文说明。由 S_{PH} 导出的协变辛形式 [7] 为

$$-\frac{1}{\kappa\gamma}\int_{\Sigma} \delta_{[1}\left(e^I \wedge e^J\right) \wedge \delta_{2]}\left(\omega_{IJ} - \frac{\gamma}{2}\varepsilon_{IJKL}\omega^{KL}\right), \tag{5}$$

where $\sum$ is any Cauchy surface and δ_1 and δ_2 are vectors tangent to the space of solutions of the resulting field equations.

其中 $\sum$ 为任意柯西曲面，δ_1 和 δ_2 是切于所得场方程解空间的向量。

A 3+1 decomposition of $\mathcal{M}$ relies on a choice of a foliation by three-dimensional surfaces. In particular, the coframes e^I should satisfy, in the dual frame $(e_0, \dots, e_3)$, that the vector fields $e_i = e_1, e_2, e_3$ are tangent to the leaves of the foliation. The leaves of this foliation are also assumed to be Cauchy surfaces of the corresponding metric tensors. On each leaf, there is an induced metric

对 $\mathcal{M}$ 做 3+1 分解需要选定一个三维曲面叶状结构。具体而言，余标架 e^I 在对偶标架 $(e_0, \ldots, e_3)$ 中应当满足: 向量场 $e_i = e_1, e_2, e_3$ 切于叶状结构的每个叶。该叶状结构的叶同时也假定是对应度量张量的柯西曲面。在每个叶上存在诱导度量

$$q = q_{ij} e^i e^j = \left(e^1\right)^2 + \left(e^2\right)^2 + \left(e^3\right)^2. \tag{6}$$

The tensor q_{ij} lowers and raises lower-case Roman indices, which range from 1 to 3 . The corresponding torsion-free connection, $\Gamma^i = \Gamma^1, \Gamma^2, \Gamma^3$, is given by 1-forms satisfying

张量 q_{ij} 负责升降取值范围为 1 到 3 的小写罗马指标。对应的无挠联络 $\Gamma^i = \Gamma^1, \Gamma^2, \Gamma^3$ 由满足以下条件的 1-形式给出:

$$de^i + \varepsilon^i{}_{jk} \Gamma^j \wedge e^k = 0,$$

where ε_{ijk} is the alternating symbol. While the extrinsic curvature can be expressed in terms of 1-forms $K^i = K^1, K^2, K^3$ defined by

其中 ε_{ijk} 是交错符号。而外曲率可以通过由下式定义的 1-形式 $K^i = K^1, K^2, K^3$ 表示

$$K_a^i := e_b^i \nabla_a n^b,$$

with n the normal to the leaves of the foliation, that is, $n = e_0$, with e_0 the timelike vector field in the frame e_I . The symplectic 2-form (5) written in terms of the $3+1$ decomposition reads

其中 n 是叶状结构叶的法向量，即 $n = e_0$ ，且 e_0 是标架 e_I 中的类时向量场。用 $3+1$ 分解写出的辛 2-形式 (5) 为

$$\frac{1}{\kappa\gamma} \int_{\Sigma} \left(\delta_1 E_i^a \delta_2 A_a^i - \delta_2 E_i^a \delta_1 A_a^i\right) d^3 x, \tag{7}$$

where

其中

$$E_i^a := \sqrt{\det q}\, e_i^a,\ A_a^i = \Gamma_a^i + \gamma K_a^i, \tag{8}$$

and the frame vectors $e_i = e_1, e_2, e_3$ are tangent to $\sum$ and dual to the coframe e^i . The symplectic form above gives rise to the Poisson brackets

且标架向量 $e_i = e_1, e_2, e_3$ 切于 $\sum$ ，对偶于余标架 e^i 。上述辛形式导出泊松括号

$$\left\{A_a^i(x), E_j^b(y)\right\} = \kappa\gamma \delta_i^j \delta_a^b \delta^{(3)}(x, y). \tag{9}$$

Clarification of the resulting symmetries is in order. The Palatini-Holst action (4) is symmetric with respect to the diffeomorphisms of $\mathcal{M}$ and local Lorentz rotations of the coframe field, which are accompanied by transformations of the connection 1-forms $\omega^I{}_J$:

下面我们来澄清所得的对称性。帕拉蒂尼-霍尔斯特作用量 (4) 关于 $\mathcal{M}$ 的微分同胚以及余标架场的局域洛伦兹旋转不变，这一过程伴随联络 1-形式 $\omega^I{}_J$ 的变换:

$$e'^I = \left(g^{-1}\right)^I_J e^J,\ \omega'^I{}_J = \left(g^{-1}\right)^I{}_K \omega^K{}_L g^L{}_J + \left(g^{-1}\right)^I{}_K dg^K{}_J.$$

The $3+1$ decomposition reduces that symmetry to the diffeomorphisms preserving the foliation of $\mathcal{M}$ and to the $SO(3)$ coframe rotations preserving the frame element e_0 , which is orthogonal to the foliation (with respect to the metric tensor (3)). In handling these symmetries, we make one more step, namely, we consider a basis $\tau_i = \tau_1, \tau_2, \tau_3 \in \mathfrak{su}(2)$ of the Lie algebra of the group SU (2) such that

3+1 分解将该对称性约化为保持 $\mathcal{M}$ 叶状结构的微分同胚，以及保持正交于叶状结构 (关于度量张量 (3)) 的标架元 e_0 的 $SO(3)$ 余标架旋转。在处理这些对称性时，我们更进一步: 我们给定群 SU (2) 的李代数的一组基 $\tau_i = \tau_1, \tau_2, \tau_3 \in \mathfrak{su}(2)$ ，满足

$$\left[\tau_i, \tau_j\right] = \varepsilon^k{}_{ij}\tau_k,\ -2\,\mathrm{Tr}\left(\tau_i\tau_j\right) = q_{ij},$$

and use it to collect the 1-forms A^i_a into an $\mathfrak{su}(2)$ -valued 1-form A , and the vector densities E^a_i into an $\mathfrak{su}(2)$ -valued vector density E ,

并利用这组基将 1 形式 A^i_a 整理为一个 $\mathfrak{su}(2)$ 值 1 形式 A ，将向量密度 E^a_i 整理为一个 $\mathfrak{su}(2)$ 值向量密度 E ,

$$A = A^i_a \tau_i \otimes dx^a,\ E = E^{ia}\tau_i \otimes \partial_i. \tag{10}$$

Now, the local rotations are represented by SU (2) -valued maps $g : \sum \to \mathrm{SU}(2)$, and

现在，局域旋转由 SU (2) 值映射 $g : \sum \to \mathrm{SU}(2)$ 表示，且

$$E' = g^{-1}Eg,\ A' = g^{-1}Ag + g^{-1}dg. \tag{11}$$

With this setup, another choice of canonical variables consistent with the limit $\gamma \to 0$ would be

在该设定下，另一个与极限 $\gamma \to 0$ 相容的正则变量选择为

$$P''^a_i := \kappa\sqrt{\mathrm{detq}}e^a_i,\ A''^i_a = K^i_a + \gamma\Gamma^i_a.$$

However, the special property of the Ashtekar-Barbero variables (8) (see [8,9]) is that A^i_a is a connection 1-form defined on an SU(2) bundle covering the orthonormal frame bundle over $\sum$.

然而，Ashtekar-Barbero 变量 (8)(参见文献 [8,9]) 的特殊性质在于，A_a^i 是定义在覆盖 $\sum$ 上标准正交标架丛的 SU(2) 丛上的联络 1 形式。

The Classical Holonomy-Flux Variables

经典全纯-通量变量

The local frame rotations (11) clearly involve non-physical degrees of freedom. What captures the geometrical (physical) degrees of freedom is the parallel transport. Consider a path $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in \sum$ and the equation

局部标架旋转 (11) 显然包含非物理自由度。平行输运才是捕捉几何 (物理) 自由度的量。考虑路径 $[\tau_0, \tau_1] \ni \tau \mapsto p(\tau) \in \sum$ 和方程

$$\frac{dh(t, t_0; p, A)}{d\tau} = -A_a(p(\tau))\, \dot{p}^a(\tau)\, h(t, t_0; p, A),\ h(t_0, t_0; p, A) = \mathrm{id}.$$

We assign to every path p the corresponding parallel transport

我们为每条路径 p 赋予对应的平行输运

$$h_p(A) \ :=\ h(t_1, t_0; p, A) \in \mathrm{SU}(2), \tag{12}$$

and with a slight abuse of language, we call this open-path holonomy just a holon-omy. Notice, that $h_p(A)$ is independent of orientation preserving reparametrizations of p; meanwhile orientation reversal induces the flip, $h_{p^{-1}} = (h_p)^{-1}$, where p^{-1} represents the same path, but with opposite orientation, and on the right, we have the inverse element in $\mathrm{SU}(2)$. An important property of the holonomies is the composition rule

用语上稍作简化，我们将这种开路径全绕直接称为全绕。注意，$h_p(A)$ 不依赖于 p 保定向的重新参数化；同时定向反转会带来变换 $h_{p^{-1}} = (h_p)^{-1}$，其中 p^{-1} 代表同一条路径但取相反定向，等式右侧是 $\mathrm{SU}(2)$ 中的逆元。全绕的一个重要性质是复合规则

$$h_{p_1 \circ \ldots \circ p_n}(A) = h_{p_1}(A) \ldots h_{p_n}(A). \tag{13}$$

On the other hand, to every oriented 2-surface $S \subset \sum$ (a submanifold) and smearing function $f : S \to \mathrm{su}(2)$, we assign the flux

另一方面，我们为每个定向二维曲面 $S \subset \sum$ (子流形) 和弥散函数 $f : S \to \mathrm{su}(2)$ 赋予通量

$$P_{S,f}(E) \ :=\ \frac{1}{2} \int E_i^a f^i \varepsilon_{abc} dx^b \wedge dx^c. \tag{14}$$

Note that the smearing function f may involve a point-dependent holonomy of A, chosen such that the integrand is invariant with respect to (11); however, one may also use smearing functions independent of A.

注意，弥散函数 f 可以包含依赖于点的 A 全绕，选择这种形式是为了让被积函数相对于 (11) 不变；当然也可以使用不依赖 A 的弥散函数。

The Poisson bracket between the holonomies and fluxes can be easily calculated and has a simple, geometrical structure. Indeed, if the path p begins or ends on the surface S and does not cross it anywhere else, then [10],

全绕和通量之间的泊松括号可以很容易计算，且结构简单、具有几何性。事实上，如果路径 p 起点或终点在曲面 S 上，且不在其他位置与曲面相交，那么根据文献 [10],

$$\{h_p, P_{S,f}\} = -\kappa\gamma\frac{\sigma(S,p)}{2}\begin{cases} h_p f(p(t_0)), & \text{if } p(t_0)\in S \\ -f(p(t_1))\,h_p, & \text{if } p(t_1)\in S, \end{cases} \tag{15}$$

where

其中

$$\sigma(S,p) = \begin{cases} 1, & \text{if } p \text{ lies above } S \\ -1, & \text{if } p \text{ lies below } S \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Here the terms above and below in the formula should be interpreted as follows. Since $\sum$ and S are both oriented, they divide the tangent space $T_x \sum$ at the intersection point $x = p \cap S$ into three sets: those tangent vectors that when appended to a positively oriented basis of S give a positively oriented basis of $\sum$, those that give a negatively oriented one, and those that are tangent to S . If a positively oriented tangent to p lies in the first set, we say that p is above S ; if it lies in the second, p is below S .

公式中“上方”和“下方”的术语解释如下: 由于 $\sum$ 和 S 都带有定向，它们将交点 $x = p \cap S$ 处的切空间 $T_x \sum$ 分为三类切向量: 与 S 的正定向基结合后得到 $\sum$ 正定向基的切向量、得到 $\sum$ 负定向基的切向量，以及本身切于 S 的切向量。如果 p 的一个正定向切向量属于第一类，我们称 p 在 S 上方；如果属于第二类，则称 p 在 S 下方。

If a path p can be decomposed into paths of one of the categories considered above, then via (13) and the Leibniz rule, the Poisson brackets (15) can be applied. On the other hand, the Poisson bracket (15) vanishes if the path p does not intersect S or if it is contained in S . A case that remains unaccounted for is a path p intersecting the surface S in infinite number of isolated points. We eliminate those by assuming a semi-analytic structure on $\sum$ [11], one that is a proper generalization of piecewise analyticity (see section ”Diffeomorphisms and Diffeomorphism Invariance”).

如果路径 p 可以分解为上述某一类路径，那么通过 (13) 和莱布尼茨规则就可以应用泊松括号 (15)。另一方面，如果路径 p 不与 S 相交，或完全包含在 S 内，泊松括号 (15) 为零。尚未讨论的情况是路径 p 在无穷多个孤立点与曲面 S 相交。我们通过假设 $\sum$ 上的半解析结构排除这种情况 [11]，该结构是分段解析性的合适推广 (参见章节“微分同胚与微分同胚不变性”)。

The Poisson bracket relations (15) have a remarkable consequence. The variables $P_{S,f}$ do not Poisson commute [12]. Due to the Jacobi identity, we have

泊松括号关系 (15) 有一个值得注意的结论: 变量 $P_{S,f}$ 不存在泊松对易关系 [12]。根据雅可比恒等式，我们有

$$\{\{P_{S_1,f_1}, P_{S_2,f_2}\}, h_p\} = -\{\{P_{S_2,f_2}, h_p\}, P_{S_1,f_1}\} - \{\{h_p, P_{S_1,f_1}\}, P_{S_2,f_2}\},$$

(17)

implying a nontrivial bracket $\{P_{S_1,f_1}, P_{S_2,f_2}\}$. Closer inspection shows that it is nonzero only for $S_1 \cap S_2 \neq \varnothing$ and has nonzero brackets with h_p only if $S_1 \cap S_2 \cap \ p \neq \varnothing$. Non-Poisson-commuting P s are surprising at first sight since canonical commutation relations would seem to imply that the field E commutes with itself. However, the non-commutativity can be understood in a natural way by defining the P s as (infinite dimensional) vector fields [12] or by the observation that

这意味着存在非平凡括号 $\{P_{S_1,f_1}, P_{S_2,f_2}\}$。进一步研究表明，它仅在 $S_1 \cap S_2 \neq \varnothing$ 时非零，且仅当 $S_1 \cap S_2 \cap \ p \neq \varnothing$ 时，与 h_p 之间存在非零括号。不对易的 P s 乍看令人惊讶，因为正则对易关系似乎意味着场 E 和自身对易。不过这种非对易性可以自然理解: 要么将 P 定义为 (无穷维) 向量场 [12]，要么注意到

$$P'_{R,f} = \frac{1}{2}\int_R d^A f^i \wedge E^a_i \varepsilon_{abc} dx^b \wedge dx^c \qquad (18)$$

for R a region with boundary $\partial R = S$ differs from $P_{S,f}$ by a term that vanishes when the Gauss constraint gets taken into account [13]. Here d^A is the covariant form derivative with respect to A . Thus, $P'_{R,f}$ will reproduce the brackets (15) while explaining (17) due to the presence of A and E .

对于带边界 $\partial R = S$ 的区域 R ，与 $P_{S,f}$ 相差一项，考虑高斯约束后该项会消失 [13]。此处 d^A 是关于 A 的协变形式导数。因此，由于 A 和 E 的存在，$P'_{R,f}$ 会重现括号 (15) 同时解释 (17)。

The holonomy variables allow better control of the local rotation transformations (11). Indeed,

环绕量变量可以更好地控制局部旋转变换 (11)。事实上,

$$h_p\left(g^{-1}Ag + g^{-1}dg\right) = g^{-1}\left(p\left(t_1\right)\right) h_p g\left(p\left(t_0\right)\right). \qquad (19)$$

In the following, we will consider functions of A depending solely on the holonomies along a finite number of paths in $\sum$. If the paths are chosen such that they form a graph Γ (i.e., they intersect in their boundaries only), then the local rotation transformations act at the vertices of Γ only. We will call such functions cylindrical [14].

在下文中，我们将考虑仅依赖于 $\sum$ 内有限条路径环绕量的 A 函数。若所选路径恰好构成图 Γ (即路径仅在边界处相交)，则局部旋转变换仅作用于 Γ 的顶点上。我们称这类函数为柱函数 [14]。

The transformation properties of the flux variables that use A -independent smearing functions are

使用与 A 无关的弥散函数时，通量变量的变换性质为

$$P_{S,f}\left(g^{-1}Eg\right) = P_{S,gfg^{-1}}\left(E\right). \tag{20}$$

That is why in some cases, we use holonomy-dependent f s [15] such that

因此在部分情形下，我们使用依赖环绕量的 f [15]，使得

$$P_{S,f}\left(g^{-1}Eg, g^{-1}Agg^{-1}dg\right) = P_{S,f}\left(A, E\right). \tag{21}$$

The Quantum Holonomy-Flux Variables

量子全纯-通量变量

Identifying a quantum theory corresponding to the classical structures discussed in the last section requires the choice of an algebra of kinematic quantities. The only hard requirement on the algebraic structure is that, to first order in $\hbar$, the Poisson relations (9) are realized as commutators, a consistency requirement for the classical limit. This still leaves a lot of possibilities. Further reasonable requirements are diffeomorphism covariance, gauge covariance, and simplicity. Diffeomorphisms of $\sum$ and gauge transformations are generated by the constraints and act on the fields (8). They can hence be expected to act on the algebra underlying the quantum theory. The algebra should be closed under these transformations. The algebraic structure should also be free of fixed classical structures, such as a fixed classical metric. This is because they would not transform, they would distinguish a (gauge or coordinate) frame, and thus they would be unnatural in a generally covariant theory.

要确定与上一节讨论的经典结构对应的量子理论，需要选择运动学量代数。对该代数结构仅有的硬性要求是: 在 $\hbar$ 一阶近似下，泊松关系 (9) 可通过对易子实现，这是经典极限的一致性要求。这仍留下了诸多可能性。进一步的合理要求包括微分同胚协变性、规范协变性与简洁性。$\sum$ 的微分同胚和规范变换由约束生成，并作用在场 (8) 上。因此可以预期它们会作用在量子理论的基础代数上。该代数在这些变换下应当封闭。代数结构还不应包含固定经典结构，例如固定经典度量。这是因为这类结构无法变换，它们会区分 (规范或坐标) 标架，因此在广义协变理论中是不自然的。

The algebra usually chosen is the holonomy-flux algebra, generated by elements

通常选取的代数是由元素生成的全纯-通量代数

$$\left(\widehat{h}_p\right)^a{}_b,\ \widehat{P}_{S,f}. \tag{22}$$

These can be thought of as representing the holonomies (12) of A along paths p in $\sum$ and fluxes,(14), where f is a triple of smearing functions and S is an oriented surface. The classical quantities transform under diffeomorphisms of $\sum$ in a simple way since the integrands are top forms on the submanifolds being integrated over. As for algebraic relations, the algebra elements inherit the relations governing the parallel transport, for example,

这些元素可以理解为表示 A 沿 $\sum$ 中路径 p 的全纯 (12)，以及通量 (14)，其中 f 是一组弥散函数，S 是一个定向曲面。经典量在 $\sum$ 的微分同胚下变换形式十分简单，因为被积函数是积分子流形上的 top 形式。至于代数关系，代数元素继承了控制平行输运的关系，例如:

$$\widehat{h}_{p_2 \circ p_1} = \widehat{h}_{p_2} \widehat{h}_{p_1}, \; \widehat{h}_p \left(\widehat{h}_p \right)^{\dagger} = \mathbb{1}, \tag{23}$$

where these equations now have to be read as matrix equations with algebra-valued entries, the dagger being matrix transpose and algebra adjoint. Similarly, the symbols $\widehat{P}_{S,f}$ inherit relations from their classical counterparts, such as

此时这些方程需要解读为 entries 为代数值的矩阵方程，dagger 代表矩阵转置与代数伴随。类似地，符号 $\widehat{P}_{S,f}$ 也从经典对应物继承了关系，例如:

$$\widehat{P}_{S,f_1+f_2} = \widehat{P}_{S,f_1} + \widehat{P}_{S,f_2}, \; \widehat{P}^{\dagger}_{S,f} = \widehat{P}_{S,\bar{f}}. \tag{24}$$

The key non-classical relation in the holonomy-flux algebra is the commutator

全纯-通量代数中关键的非经典关系是对易子

$$\left[P_{S,f}, h_p\right] = \begin{cases} 0 & \text{if } S \cap p = \varnothing \\ 8\pi \ell_{\mathrm{P}}, \sigma(S,p)\, h_2 \tau_i f^i(x)\, h_1 & \text{otherwise,} \end{cases} \tag{25}$$

mirroring (15). Here x is a single intersection point of S and p and σ is the sign (16) depending on the relative orientation of p and S . If p is tangential to S at x, σ vanishes. There is a straightforward generalization to the case of multiple intersections where the result is a sum over intersections, with each intersection a term like (25). Note that there is closure under (25) since the commutator yields a sum of products of holonomy matrix elements. The algebra generated by the relations (23), (24), and (25), together with the Jacobi identity, is called the holonomy-flux algebra $\mathfrak{A}_{\mathrm{HF}}$ (For a careful definition of $\mathfrak{A}_{\mathrm{HF}}$ see, for example,[11]. There are slightly different definitions of this algebra in the literature, depending on whether one wants to impose additional higher-order commutator relations (see [16, 17] for more details).).

它是 (15) 的镜像。此处 x 是 S 与 p 的单个交点，σ 是取决于 p 与 S 相对取向的符号 (16)。若 p 在 x, σ 处与 S 相切，该结果为零。对于多个交点的情况可以直接推广，结果为所有交点的求和，每个交点对应一项如 (25) 的形式。注意 (25) 下代数是封闭的，因为对易子给出全纯矩阵元的乘积和。由关系 (23)、(24)、(25) 结合雅可比恒等式生成的代数称为全纯-通量代数 $\mathfrak{A}_{\mathrm{HF}}$ (例如参见文献 [11] 了解 $\mathfrak{A}_{\mathrm{HF}}$ 的严谨定义。文献中对该代数的定义略有不同，取决于是否需要施加额外的高阶对易关系，更多细节参见 [16, 17]。)。

Note that (25) only uses ingredients such as intersection, evaluation, and relative orientation that are invariant under diffeomorphisms. This implies that diffeomorphisms act as algebra homomorphisms on $\mathfrak{A}_{\mathrm{HF}}$.

注意 (25) 仅使用交点、赋值、相对取向这些在微分同胚下不变的要素。这意味着微分同胚在 $\mathfrak{A}_{\mathrm{HF}}$ 上以代数同态的方式作用。

Another important aspect of (25) and $\mathfrak{A}_{\mathrm{HF}}$ is that non-commutativity of spatial geometry is unavoidable. As anticipated by the classical relations (17), one finds

(25) 与 $\mathfrak{A}_{\mathrm{HF}}$ 的另一个重要性质是空间几何的非对易性不可避免。正如经典关系 (17) 所预示的，我们有

$$\left[\left[\widehat{P}_{S_1,f_1},\widehat{P}_{S_2,f_2}\right],\widehat{h}_p\right]=-\left[\left[\widehat{P}_{S_2,f_2},\widehat{h}_p\right],\widehat{P}_{S_1,f_1}\right]-\left[\left[\widehat{h}_p,\widehat{P}_{S_1,f_1}\right],\widehat{P}_{S_2,f_2}\right], \tag{26}$$

which implies nontrivial commutators of $\widehat{P}$ s in general. Since these objects correspond to the spatial metric geometry, one is dealing with a form of non-commutative Riemannian geometry.

一般而言它意味着 $\widehat{P}$ s 的非平凡对易子。由于这些量对应空间度量几何，我们研究的是一类非对易黎曼几何。

Now representations of this algebra can be studied. One important representation is the Ashtekar-Lewandowski representation [1, 14, 18] to which we turn next.

现在我们可以研究该代数的表示。接下来我们将介绍一个重要表示: 阿西特卡-莱万多夫斯基表示 [1, 14, 18] 。

Ashtekar-Lewandowski Representation

阿西特卡-莱万多夫斯基表示

To obtain the Ashtekar-Lewandowski representation, one can follow the traditional strategy, split the variables (A, E) into ”positions” A and ”momenta” E , and define quantum states to be functions of the positions. It will turn out that the result is unique after some assumptions. For this purpose, we use the algebra $\mathrm{Cyl}^{(\infty)}$ of cylindrical functions, that is, the functions that can be written in the following way:

要得到阿西特卡-莱万多夫斯基表示，可遵循传统策略: 将变量 (A, E) 拆分为“位置” A 和“动量” E ，并将量子态定义为位置的函数。在若干假设下，最终结果是唯一的。为此我们使用柱函数的代数 $\mathrm{Cyl}^{(\infty)}$ ，柱函数即可以写成如下形式的函数:

$$\Psi(A)=\psi\left(h_{p_1}(A),\ldots,h_{p_n}(A)\right), \tag{27}$$

where $p_1,\ldots,p_n$ are arbitrary paths in $\sum$, the number n depends on Ψ and is arbitrary, and $\psi\in C^{\infty}\left(\mathrm{SU}(2)^n\right)$. Clearly, these functions define a subalgebra of the algebra of all the functions of the variable A . Given a cylindrical function Ψ , the set of the paths in (27) is not unique. For example, we can always subdivide a path into two or change its orientation or just add a new, unnecessary path. The key observation is that two semi-analytic paths can intersect one another only at a finite set of isolated points and along a finite set of connected segments. As a consequence, given a cylindrical function (27), we can subdivide the paths in such a manner that the resulting paths intersect each other at most at one or at two endpoints, that is, they describe a graph embedded in $\sum$. Thus, every cylindrical function can be written in the form (27), where the paths $p_1,\ldots,p_n$ are edges of a graph Γ embedded in $\sum$.

其中 $p_1, \ldots, p_n$ 是 $\sum$ 内任意路径，数目 n 由 Ψ 决定且任意，$\psi \in C^\infty\left(\mathrm{SU}(2)^n\right)$。显然，这些函数构成了变量 A 全函数代数的一个子代数。对于给定的柱函数 Ψ，(27) 式中的路径集合不唯一。例如，我们总能将一条路径拆分为两条，改变其方向，或是直接新增一条多余的路径。核心结论是: 两条半解析路径仅能在有限个孤立点相交，或是沿有限段连通弧相交。因此，对于给定的柱函数 (27)，我们总能拆分路径，使得拆分后的路径最多只在一个或两个端点处相交，也就是说这些路径构成了一张嵌入 $\sum$ 的图。由此，任意柱函数都可以写成 (27) 的形式，且其中路径 $p_1, \ldots, p_n$ 是嵌入 $\sum$ 的图 Γ 的边。

To define the proto-Hilbert product between two states Ψ and Ψ', we choose a graph Γ, specified by the paths $\{p_1, \ldots, p_n\}$, such that each state can be written in the form (27), and then define

为了定义两个量子态 Ψ 和 Ψ' 之间的原希尔伯特内积，我们选取一张由路径 $\{p_1, \ldots, p_n\}$ 指定的图 Γ，使得两个态都可以写成 (27) 的形式，随后定义:

$$(\Psi, \Psi') := \int dg_1 \ldots dg_n \overline{\psi(g_1, \ldots, g_n)} \psi'(g_1, \ldots, g_n). \tag{28}$$

Importantly, the result is independent of the choice of graph. The Hilbert space $\mathcal{H}_{\mathrm{AL}}$ of the quantum states is defined to be the completion of Cyl in the norm defined by the product $(\cdot, \cdot)$.

重要的是，该结果与图的选取无关。量子态的希尔伯特空间 $\mathcal{H}_{\mathrm{AL}}$ 被定义为 Cyl 在由内积 $(\cdot, \cdot)$ 诱导范数下的完备化。

Every cylindrical function Ψ is promoted to a quantum operator $\hat{\Psi}$ acting in $\mathcal{H}$ naturally: $\hat{\Psi}\Psi' = \Psi\Psi'$. While every flux observable $P_{S,f}$ gives rise to a quantum operator $\hat{P}_{S,f}$ defined (originally) on $\mathrm{Cyl}^{(\infty)}$ by $\hat{P}_{S,f}\Psi := i\hbar\left\{P_{S,f}, \Psi\right\}$. The properties of the ”position” operators are obvious. The flux operator is symmetric (and even essentially self-adjoint), provided the holonomies potentially involved in the definition of f are all contained in the surface S and the function f is real valued.

任意柱函数 Ψ 都可以自然地提升为作用在 $\mathcal{H}$ 上的量子算符 $\hat{\Psi}$: $\hat{\Psi}\Psi' = \Psi\Psi'$。而任意通量可观测量 $P_{S,f}$ 都对应一个量子算符 $\hat{P}_{S,f}$，它 (最初) 定义在 $\mathrm{Cyl}^{(\infty)}$ 上，表达式为 $\hat{P}_{S,f}\Psi := i\hbar\left\{P_{S,f}, \Psi\right\}$。“位置”算符的性质十分显然。若 f 定义中涉及的全同胚都包含在曲面 S 内，且函数 f 是实值函数，则通量算符是对称的 (甚至本质自伴)。

The flux operator gives rise to a quantum spin operator $\hat{J}_i^{x[p]}$ assigned to a point $x \in \sum$ and a class $[p]$ of curves that begin at x and share initial segments. Given a cylindrical function Ψ, we write it in the form (27) with a graph such that one of the edges, say p_1, begins at x and belongs to the class $[p]$. Then,

通量算符对应了分配给点 $x \in \sum$ 与起始于 x、共享初始段的曲线类 $[p]$ 的量子自旋算符 $\hat{J}_i^{x[p]}$。给定柱函数 Ψ，我们将其按 (27) 的形式写在一张图上，假设其中一条边为 p_1，它起始于 x 且属于曲线类 $[p]$，那么:

$$\hat{J}_i^{x[p]}\Psi(A) = i\hbar \frac{d}{ds}_{|s=0} \psi\left(h_{p_1}(A) e^{s\tau_i}, h_{p_2}(A), \ldots, h_{p_n}(A)\right). \tag{29}$$

In terms of the quantum spin operators, the quantum flux operator is

用量子自旋算符表示，量子通量算符为

$$\widehat{P}_{S,f}=\frac{\kappa\gamma}{2}\sum_{x\in S}f^i(x)\left(\sum_{[p]\text{ going up}}\hat{J}_i^{x,p}-\sum_{[p]\text{ going down}}\hat{J}_i^{x,p}\right)$$
$$=\frac{\kappa\gamma}{2}\sum_{x\in S}f^i(x)\left(\hat{J}_i^{x,S\uparrow}-\hat{J}_i^{x,S\downarrow}\right). \quad (30)$$

The sum on x ranges over the full surface S , and that over $[p]$ ranges over all the (classes) of edges at x . However, when the operator is applied to a cylindrical function, then these sums only range over the isolated intersections of the surface with the curves the function depends on and the edges $[p]$ that overlap the curves of the cylindrical function.

对 x 的求和遍历整个曲面 S ，对 $[p]$ 的求和遍历 x 处所有边的 (等价类)。但当该算符作用于柱函数时，求和仅遍历曲面与柱函数依赖的曲线的孤立交点，以及与柱函数的曲线重叠的边 $[p]$ 。

Diffeomorphism $\phi:\sum\rightarrow\sum$ naturally act on the cylindrical functions (27) via

微分同胚 $\phi:\sum\rightarrow\sum$ 通过以下方式自然作用于柱函数 (27)

$$U_\phi\Psi(A)=\Psi(\phi^*A)=\psi\left(h_{\phi(p_1)}(A),\ldots,h_{\phi(p_n)}(A)\right).$$

The operator U_ϕ is unitary in $\mathcal{H}$. The fluxes are also diffeomorphism covariant

算符 U_ϕ 在 $\mathcal{H}$ 中是幺正的。通量也满足微分同胚协变性

$$U_\phi\widehat{P}_{S,f}U_\phi^{-1}=\widehat{P}_{\phi^{-1}(S),\phi^*f},$$

and if f involves holonomies along paths $q_1,\ldots,q_k$, then ϕ^*f depends in the same way on the holonomies along $\phi^{-1}q_1,\ldots,\phi^{-1}q_n$. It should be noted, however, that diffeomorphisms do not admit infinitesimal generators acting in the space of the cylindrical functions.

若 f 包含路径 $q_1,\ldots,q_k$ 上的全纯性，则 ϕ^*f 以相同的方式依赖于 $\phi^{-1}q_1,\ldots,\phi^{-1}q_n$ 上的全纯性。但需要注意，微分同胚不存在作用在柱函数空间上的无穷小生成元。

The local rotations (11) also act naturally and unitarily in the Hilbert space $\mathcal{H}$ via (19). They are generated by the Lie algebra of the local rotation operators

局域旋转 (11) 也通过 (19) 在希尔伯特空间 $\mathcal{H}$ 中自然幺正作用。它们由局域旋转算符的李代数生成

$$\int_\sum d^3x\left(\left(\widehat{D_a\Lambda}\right)E^a\right)=\frac{\kappa\gamma}{2}\Lambda^i(x)\sum_{x\in\sum}\sum_{[p]\text{ at }x}\hat{J}_i^{x,[p]}=:\frac{\kappa\gamma}{2}\Lambda^i(x)\sum_{x\in\sum}\hat{J}_i^{x,\text{ tot}}.$$

(31)

The compact notation of the final expression relates the total spin operator, just introduced, to the total spin operators $S\uparrow,S\downarrow$, and $S\parallel$, which accounts for the curves contained in S , via

最终表达式的紧凑记号通过下式将刚引入的总自旋算符，与描述包含在 S 中曲线的总自旋算符 $S\uparrow$,$S\downarrow$ 和 $S\parallel$ 联系起来

$$\hat{J}_i^{x,\,\mathrm{tot}} = \hat{J}_i^{x,S\uparrow} + \hat{J}_i^{x,S\downarrow} + \hat{J}_i^{x,S\parallel}. \tag{32}$$

The algebra generated by the cylindrical functions and the flux operators fulfills the relations of the holonomy-flux algebra $\mathfrak{A}_{\mathrm{HF}}$ and, thus, forms a representation of $\mathfrak{A}_{\mathrm{HF}}$ on the Hilbert space $\mathcal{H}$. Moreover, spatial diffeomorphisms act in a unitary manner as described above and leave the Ashtekar-Lewandowski vacuum, represented by the constant cylindrical function, invariant. It can be shown that this is the only representation of $\mathfrak{A}_{\mathrm{HF}}$ in which the spatial diffeomorphisms are unitarily represented and that contains an invariant cyclic vector [11, 19].

由柱函数和通量算符生成的代数满足全纯-通量代数 $\mathfrak{A}_{\mathrm{HF}}$ 的关系，因此构成了 $\mathfrak{A}_{\mathrm{HF}}$ 在希尔伯特空间 $\mathcal{H}$ 上的一个表示。此外，空间微分同胚按上述方式幺正作用，且保持由常数柱函数表示的阿西特卡-莱万多夫斯基真空不变。可以证明，这是 $\mathfrak{A}_{\mathrm{HF}}$ 唯一满足空间微分同胚被幺正表示、且包含不变循环向量的表示 [11, 19]。

If we fix a graph Γ , specified by paths $p_1, \dots, p_n \subset M$, in the definition of a cylindrical function, then the Peter-Weyl theorem provides an orthonormal basis. Fix an orthonormal basis in the Hilbert space $\mathcal{H}_{j_I}$ of irrep j_I ; then a basis element is defined by assigning to each edge p_I the $\left(n^I, m_I\right)$ -th entry of the Wigner matrix $D_{(j_I)}{}^{m_I}{}_{m'_I}$. From this data, we construct a function on $\mathrm{SU}(2)^n$,

如果我们在柱函数的定义中固定一个由路径 $p_1, \dots, p_n \subset M$ 指定的图 Γ，彼得-外尔定理可以给出一组标准正交基。在不可约表示 j_I 的希尔伯特空间 $\mathcal{H}_{j_I}$ 中固定一组标准正交基，之后通过给每条边 p_I 分配维格纳矩阵 $D_{(j_I)}{}^{m_I}{}_{m'_I}$ 的第 $\left(n^I, m_I\right)$ 个元来定义一个基元。我们可以从这些数据出发，构造出 $\mathrm{SU}(2)^n$ 上的一个函数,

$$\psi(g_1, \dots, g_n) = \sqrt{(2j_1+1)\dots(2j_n+1)}D_{(j_1)}{}^{m_1}{}_{m'_1}(g_1)\dots D_{(j_n)}{}^{m_n}{}_{m'_n}(g_n).$$

(33)

In order to control the properties of that function with respect to the gauge transformations, at each intersection point v between incoming edges $p_{I_1}, \dots, p_{I_k}$ and outgoing, say $p_{J_1}, \dots, p_{J_l}$, we introduce tensors $\iota_{v m'_{I_1}\dots m'_{I_k}} m_{J_1} \dots m_{J_l}$, termed intertwiners, which intertwine the corresponding representations (or the dual ones in the case of the outgoing edges) into a new irreducible representation j_v (Let $D_\otimes = D_{(I_1)} \otimes \dots \otimes D_{(I_k)} \otimes D^*_{(J_1)} \otimes \dots \otimes D^*_{(J_l)}$ is the tensor product representation. The intertwiner $\iota_{v m'_{I_1}\dots m'_{I_k}}{}^{m_{J_1}\dots m_{J_l}}{}_{m_1\dots m_{I_l}}$ is given by the matrix elements of an equivariant map

为了控制该函数关于规范变换的性质，我们在入射边 $p_{I_1}, \dots, p_{I_k}$ 与出射边 (例如 $p_{J_1}, \dots, p_{J_l}$) 的每个交点 v 处引入张量 $\iota_{v m'_{I_1}\dots m'_{I_k}} m_{J_1} \dots m_{J_l}$ ，称为交缠子，它将相应表示 (出射边对应对偶表示) 交缠为一个新的不可约表示 j_v (设 $D_\otimes = D_{(I_1)} \otimes \dots \otimes D_{(I_k)} \otimes D^*_{(J_1)} \otimes \dots \otimes D^*_{(J_l)}$ 为张量积表示，交缠子 $\iota_{v m'_{I_1}\dots m'_{I_k}}{}^{m_{J_1}\dots m_{J_l}}{}_{m_1\dots m_{I_l}}$ 由等变映射的矩阵元给出

$$\iota : \mathcal{H}_\otimes \to \mathcal{H}_{j_v},\ D_{(j_v)} \circ \iota = \iota \circ D_\otimes.$$

The final m index of ι labels the states of $\mathcal{H}_{j_v}$; below we will also consider gauge-invariant states with $j_v = 0$, and this index will not always appear.), and contract them correspondingly,

ι 的最终 m 指标标记 $\mathcal{H}_{j_v}$ 的态；下文我们也会考虑带有 $j_v = 0$ 的规范不变态，该指标不总会出现)，随后对它们做相应缩并，

$$\otimes_v \iota_{v_\alpha} \lrcorner \otimes_I \sqrt{2j_I + 1} D_{(j_I)}. \tag{34}$$

In particular, if we take intertwiners into the trivial representation only, the SU(2) invariants, then we obtain a gauge-invariant cylindrical function, also referred to as a spin network [20].

特别地，如果我们仅将交缠子取为平凡表示，即 SU(2) 不变量，就会得到一个规范不变的柱函数，也称为自旋网 [20]。

In fact, since any operator in one of the families $\left\{\hat{J}_i^{y,[p]}\hat{J}^{iy,[p]}\right\}_{y,[p]}$ and $\left\{\hat{J}^{y,\,\mathrm{tot}}\right\}_y$ commutes with any other one in these families, there exists a joint eigenbasis for all of them. This basis consists of the states (34). It follows, in particular, that an operator $\widehat{O}$ satisfying

事实上，由于族 $\left\{\hat{J}_i^{y,[p]}\hat{J}^{iy,[p]}\right\}_{y,[p]}$ 与 $\left\{\hat{J}^{y,\,\mathrm{tot}}\right\}_y$ 中任意算子都与这两个族内的其他所有算子对易，因此存在所有算子的共同本征基。该基由态 (34) 构成，由此可知，满足条件的算子 $\widehat{O}$

$$\left[\widehat{O}, \hat{J}_i^{y,[p]}\hat{J}^{iy,[p]}\right] = 0 = \left[O, \hat{J}^{y,\,\mathrm{tot}}\right] \quad \text{for all } y, [p], \tag{35}$$

must be diagonal in this basis as well.

也必定在该基下对角。

Diffeomorphisms and Diffeomorphism Invariance

微分同胚与微分同胚不变性

In LQG, we distinguish between many types of diffeomorphisms and smoothness. In this subsection, we discuss the semi-analytic category and the corresponding diffeomorphisms with respect to which the quantum theory presented above is invariant. Later, in section ”Other Phases of Quantum Geometry”, we mention other directions of theory construction based on other diffeomorphisms, which have not been fully exploited, although they have provided interesting and sometimes extremely mathematically sophisticated results.

在圈量子引力 (LQG) 中，我们区分多种类型的微分同胚与光滑性。本小节我们讨论半解析范畴与相应的微分同胚，上文介绍的量子理论对这类微分同胚是不变的。之后，我们会在“量子几何的其他相”一节提及基于其他微分同胚构造理论的其他方向，这些方向尚未得到充分开发，但已经产出了有趣、有时在数学上极为精妙的结果。

In one dimension, a semi-analytic function $f : \mathbb{R} \to \mathbb{R}$ is any differentiable function that is piecewise analytic. To generalize this definition to higher dimensions, a suitable notion of ”piecewise” is required. The idea invented for the rigorous formulation of LQG diffeomorphism invariance was to introduce a new category of differentiability through the theory of semi-analytic sets and semi-analytic partitions [11, 21, 22] . If a function defined on $\mathbb{R}^n$ is analytic when restricted to every element of some semi-analytic partition of $\mathbb{R}^n$, then we call it semi-analytic. The local version of this property provides the general definition [11]. Notice that every semi-analytic partition of $\mathbb{R}^n$ contains open subsets as well as subsets of lower dimensions, for example, cube interiors, face interiors, site interiors, and vertices. The family of real-valued semi-analytic functions defined in $\mathbb{R}^n$ has all the properties needed to define semi-analytic manifolds and submanifolds, as well as semi-analytic diffeomorphisms [11]. The semi-analytic category combines important properties of the analytic category on the one hand and differentiable (or smooth) category on the other. In the semi-analytic category, every finite family of curves contained in a manifold M can be cut into finitely many pieces that fix an embedded graph in M . Given a 2-surface S in a 3-manifold, every curve can be cut into finitely many pieces such that each of the pieces either does not intersect S or intersects at a single point or overlaps S . These properties are adopted from the analytic category. On the other hand, local (semi-analytic) diffeomorphisms still exist for arbitrarily small compact supports, as in the case of the differentiable category.

一维情况下，半解析函数 $f : \mathbb{R} \to \mathbb{R}$ 是任意分段解析的可微函数。要将这个定义推广到更高维，需要一个合适的“分段”概念。为严格表述 LQG 微分同胚不变性而提出的思路是，通过半解析集和半解析分划理论 [11, 21, 22] 引入一种新的可微性范畴。如果定义在 $\mathbb{R}^n$ 上的函数，限制在 $\mathbb{R}^n$ 某个半解析分划的每个元素上都是解析的，我们就称其为半解析函数。该性质的局部形式给出了通用定义 [11]。注意，$\mathbb{R}^n$ 的每个半解析分划既包含开子集，也包含低维子集，例如立方体内部、面内部、棱内部和顶点。定义在 $\mathbb{R}^n$ 上的实值半解析函数族具备定义半解析流形、子流形以及半解析微分同胚所需的全部性质 [11]。半解析范畴一方面结合了解析范畴的重要性质，另一方面结合了可微 (光滑) 范畴的重要性质。在半解析范畴中，流形 M 内的任意有限曲线族都可以分割为有限段，这些段共同确定了嵌入 M 中的一个图。给定 3-流形中的 2-曲面 S ，任意曲线都可以分割为有限段，每一段要么不与 S 相交，要么仅在单点相交，要么完全重合于 S 的一部分。这些性质都来自解析范畴。另一方面，和可微范畴的情况一样，半解析范畴仍然存在支撑在任意小紧集上的局部 (半解析) 微分同胚。

The quantum holonomy and quantum flux operators defined above are diffeomorphism covariant, that is, they are unitarily mapped by diffeomorphisms into other holonomy and flux operators. In contrast, the total quantum volume operator, or the integral of the quantum scalar curvature operator, is diffeomorphism invariant. What are other diffeomorphism invariant operators? It turns out that every self-adjoint diffeomorphism invariant operator that contains all the cylindrical functions in its domain preserves the graphs: a cylindrical function defined by using a given graph is mapped into a cylindrical function that can be defined with the same graph [1]. This theorem underlies ”algebraic LQG” [23-26].

上文定义的量子环绕度 (量子全纯) 算子与量子通量算子是微分同胚协变的，即它们会被微分同胚通过幺正映射变换为其他的环绕度与通量算子。与之相对，总量子体积算子，或者量子标曲率算子的积分，是微分同胚不变的。还有哪些微分同胚不变算子？结论是: 定义域包含所有柱函数的任意自伴微分同胚不变算子都保图: 利用给定图定义的柱函数会被映射为可由该图定义的柱函数 [1]。该定理是“代数量子引力 (代数 LQG)”的基础 [23-26]。

The diffeomorphically invariant characterization of an immersed graph includes its global and local properties. The former are the generalized knots and links formed by graph edges. The latter describe how edges meet at vertices. Every generic triple of edges meeting at a vertex can be diffeomorphically transformed into intersection of the three axes of any fixed coordinate system. If there is a fourth edge, the remaining freedom to scale the axis can be used to fix the position of the edge arbitrarily inside a certain region representing one-eighth of the angle range. Then, a location of a fifth edge meeting at this vertex (if it is there) becomes a diffeomorphism invariant feature. In this manner, starting from valency five, graph vertices contribute a continuous set of diffeomorphism invariant degrees of freedom (see [27]).

浸入图的微分同胚不变刻画包含整体性质与局部性质。整体性质是图的边形成的广义扭结与环绕，局部性质描述边如何在顶点交汇。任意交汇于一个顶点的三条一般边都可以通过微分同胚变换为任意固定坐标系三个坐标轴的相交形式。如果存在第四条边，剩余的坐标轴缩放自由度可以用来将这条边固定在代表八分之一角度范围的特定区域内的任意位置。那么，交汇于该顶点的第五条边 (如果存在) 的位置就成为了一个微分同胚不变特征。按照这种方式，从五价顶点开始，图顶点贡献了连续的微分同胚不变自由度 (参见 [27])。

Quantum Geometry

量子几何

The states of the Ashtekar-Lewandowski representation are states of extrinsic and intrinsic geometry of the spatial slice $\sum$. This can be seen most directly when probing them with operators representing various aspects of this geometry. In the present chapter, we will restrict ourselves to the intrinsic Riemannian geometry of the spatial slice and consider the corresponding operators and their properties in sections ”The Area Operators”, ”The Angle Operator”, ”The Volume Operators”, ”The Inverse Metric Tensor Operator”, ”The Length Operator”, and ”The Ricci Scalar Operator à la Regge”. What emerges is a geometry that associates the vertices of spin networks with the volume of spatial regions and the edges with areas.

阿西特卡-莱万多夫斯基表示的态是空间切片 $\sum$ 的外禀和内禀几何的态。我们用表征该几何各方面的算符对这些态探测后，就能最直接地看出这一点。在本章中，我们将仅探讨空间切片的内禀黎曼几何，并在“面积算符”“角度算符”“体积算符”“逆度量张量算符”“长度算符”和“里奇标量算符: 基于里奇方法”这些小节中讨论对应的算符及其性质。最终得到的几何是: 自旋网络的顶点对应空间区域的体积，自旋网络的边对应面积。

Somewhat surprisingly, the quantum geometry obtained through the study of general relativity can also be seen as the quantization of piecewise flat discrete geometries. In section ”Polyhedral States”, we briefly explain how this picture arises in loop quantum gravity. This also lays groundwork for section ”Discrete Geometry” of this chapter, where quantum discrete geometries will be developed starting from their classical foundations.

有些出人意料的是，通过研究广义相对论得到的量子几何也可以看作分片平坦离散几何的量子化。在“多面体态”小节中，我们会简要解释该图景如何在圈量子引力中产生，这也为本章“离散几何”小节奠定基础，我们将在该小节从经典基础出发构建量子离散几何。

In section ”Quantum-Reduced Gravity”, we discuss a certain limit of the quantum geometric states and operators that simplifies calculations and can be used in quantum cosmology. Finally, section ”Other Phases of Quantum Geometry” briefly presents some extensions of the picture of quantum geometry in the presence of matter, in other representations of holonomies and fluxes, and in some extensions of the entire formalism.

在“量子约化引力”小节中，我们会讨论量子几何态与算符的一种极限，该极限可以简化计算，可用于量子宇宙学。最后，“量子几何的其他相”小节简要介绍了量子几何图景的若干推广，涵盖存在物质的情况、全同与通量的其他表示，以及整个形式体系的部分扩展。

The Area Operators

面积算符

The first geometric operator to be defined in loop quantum gravity, and perhaps the simplest, is the area operator [10, 28]. It is very natural in this setting because the area 2-form is essentially the length (in internal space) of the field E [29].

面积算符是圈量子引力中第一个被定义的几何算符，或许也是最简单的一个 [10, 28]。它在该框架中十分自然，因为面积 2-形式本质上就是场 E 在内空间中的长度 [29]。

Consider a two-dimensional surface $S \subset \sum$. For each classical frame density field E_i^a , cf. (8), the area element d^2s in S can be approximated by the fluxes of (14) through small pieces S_n that set a partition of S . Indeed, for every function F defined on S ,

考虑二维曲面 $S \subset \sum$。对每个经典标架密度场 E_i^a ，参见式 (8)，S 中的面积元 d^2s 可以通过 (14) 对划分 S 的小单元 S_n 的通量近似得到。确实，对定义在 S 上的任意函数 F ，

$$\int_S F d^2 s = \lim_{N\to\infty} \kappa\gamma \sum_{n=1}^{N} F\left(x_n\right) \sqrt{P_{S_n,\tau_i} P_{S_n,\tau_i} q^{ij}},\ x_n \in S_n, \tag{36}$$

provided in the limit the partition gets uniformly finer. In the quantum theory, we replace the classical fluxes by the quantum flux operators and derive the following integral in the quantum theory [10]:

该式在划分一致精细化的极限下成立。在量子理论中，我们将经典通量替换为量子通量算符，在量子理论中推导出如下积分 [10]:

$$\int_S F \widehat{d^2 s} = \frac{\kappa\gamma}{2} \sum_{x\in S} F\left(x\right) \sqrt{-\Delta_{S,x}}, \tag{37}$$

where

其中

$$-\Delta_{S,x} \ := \ q^{ij} \left(\hat{J}_i^{xS\uparrow} - \hat{J}_i^{xS\downarrow}\right)\left(\hat{J}_j^{x,S\uparrow} - \hat{J}_j^{x,S\downarrow}\right). \tag{38}$$

As in the case of the flux operator, x ranges over the points of S ; however, when the operator is applied to a cylindrical function, then only isolated intersections of the surface with the curves that the function depends on contribute. Remarkably, the operator is well defined: no infinities appear.

与通量算符的情况相同，x 遍历 S 上的点；然而，当该算符作用于柱函数时，只有曲面与函数依赖的曲线的孤立相交点会有贡献。值得注意的是，该算符是良好定义的: 不会出现无穷大。

The eigenstates and eigenvalues of the operators $-\Delta_{S,x}$ follow from the algebraic properties of the spin operators, they are

算符 $-\Delta_{S,x}$ 的本征态和本征值可由自旋算符的代数性质得到，它们是

$$\lambda = \hbar^2 \left(2j^{\uparrow}\left(j^{\uparrow}+1\right)+2j^{\downarrow}\left(j^{\downarrow}+1\right)-j^{\downarrow\uparrow}\left(j^{\downarrow\uparrow}+1\right)\right), \tag{39}$$

where $j^{\uparrow}, j^{\downarrow} = 0, \frac{1}{2}, \ldots$, and $j^{\downarrow\uparrow} = |j^{\uparrow}-j^{\downarrow}|, |j^{\uparrow}-j^{\downarrow}|+1, \ldots, |\ j^{\uparrow}+\ j^{\downarrow}\ |$, and we have used the same notation as in the definition of the flux operator (Eq. (30)).

其中 $j^{\uparrow}, j^{\downarrow} = 0, \frac{1}{2}, \ldots$，和 $j^{\downarrow\uparrow} = |j^{\uparrow}-j^{\downarrow}|, |j^{\uparrow}-j^{\downarrow}|+1, \ldots, |\ j^{\uparrow}+\ j^{\downarrow}\ |$，我们使用了与通量算符定义 (式 (30)) 相同的记号。

The eigenvalues of the area operator,

面积算符的本征值,

$$\hat{A}_S = \int_S d^2 s = \frac{\kappa\gamma}{2}\sum_{x\in S} F(x)\sqrt{-\Delta_{S,x}},$$

are all finite sums of the elementary areas $a_S = \frac{\kappa\gamma}{2}\sum_v \sqrt{\lambda_v}$, where each λ_v is of the form (39) [10].

都是基本面积 $a_S = \frac{\kappa\gamma}{2}\sum_v \sqrt{\lambda_v}$ 的有限和，其中每个 λ_v 都形如 (39) [10]。

A generic case of the eigenvalue (39) is at the intersection of a surface S with a transversal curve that is cut at the intersection point and divided into two edges, both oriented to be outgoing. Then $j^{\uparrow} = j^{\downarrow} = j$ and $j^{\downarrow\uparrow} = 0$, and the corresponding eigenvalue of the area is $a_s = \kappa\gamma\hbar\sqrt{j(j+1)}$. The simplest case, on the other hand, is a single outgoing edge, which amounts to $j^{\uparrow} = j = j^{\downarrow\uparrow}, j^{\downarrow} = 0$, and $a_S = \frac{1}{2}\kappa\gamma\hbar\sqrt{j(j+1)}$.

本征值 (39) 的一般情形是: 曲面 S 与一条横截曲线相交于一点，该曲线在交点处被分为两条都向外取向的边。此时 $j^{\uparrow} = j^{\downarrow} = j$ 和 $j^{\downarrow\uparrow} = 0$，对应的面积本征值为 $a_s = \kappa\gamma\hbar\sqrt{j(j+1)}$。另一方面，最简单的情形是单条外定向边，对应 $j^{\uparrow} = j = j^{\downarrow\uparrow}, j^{\downarrow} = 0$，且 $a_S = \frac{1}{2}\kappa\gamma\hbar\sqrt{j(j+1)}$.

If S is a connected closed surface and splits $\sum$ into two disconnected parts and if we consider only gauge-invariant cylindrical functions, then there are additional constraints, namely,

若 S 是连通闭曲面，将 $\sum$ 分为两个不连通部分，且我们只考虑规范不变的柱函数，则存在额外约束，即

$$\sum_v j_v^{\uparrow} \in \mathbb{N}, \text{ and } \sum_v j_v^{\downarrow} \in \mathbb{N}.$$

The quantum area operator is invariant under local rotations, that is, $\left[\hat{A}_S, \hat{J}_i^{x,\,\mathrm{tot}}\right] = 0$, and it is covariant with respect to diffeomorphisms: $U_\phi \hat{A}_S U_\phi^{(-1)} = \hat{A}_{\phi^{-1}A}$. The operator $\hat{q}_x$ also commutes with it, $\left[\hat{A}_S, \hat{J}_i^{y,[p]} \hat{J}^{iy,[p]}\right] = 0$.

量子面积算符在局域旋转下不变，即 $\left[\hat{A}_S, \hat{J}_i^{x,\,\mathrm{tot}}\right] = 0$ ，并且它关于微分同胚是协变的: $U_\phi \hat{A}_S U_\phi^{(-1)} = \hat{A}_{\phi^{-1}A}$ 。算符 $\hat{q}_x$ 也与它对易，$\left[\hat{A}_S, \hat{J}_i^{y,[p]} \hat{J}^{iy,[p]}\right] = 0$ 。

In some LQG models with boundary, each 2-surface is equipped with a distinguished $\mathfrak{su}(2)$ -valued function $r : S \to \mathrm{su}(2)$, and only the E_i^a frames are allowed, such that the vector field $n^a := r^i E_i^a / \sqrt{\det E}$ is normal to S . Then, we define another area operator [30],

在一些带边界的 LQG 模型中，每个二维曲面上都配有一个特殊的 $\mathfrak{su}(2)$ 值函数 $r : S \to \mathrm{su}(2)$ ，且仅允许 E_i^a 标架，满足向量场 $n^a := r^i E_i^a / \sqrt{\det E}$ 垂直于 S 。随后我们定义另一个面积算符 [30],

$$\hat{A'}_S := \int_S \left| r^i E_i^a \varepsilon_{abc} dx^b \wedge dx^c \right| = \frac{\kappa\gamma}{2} \sum_x \left| \hat{J}_i^{x,S\uparrow} r^i + \hat{J}_i^{x,S\downarrow} r^i \right| .$$

The advantage of this operator is its equidistant spectrum: the eigenvalues are 0 and $\frac{\kappa\gamma}{2} N$, with $N \in \mathbb{N}$. This is compatible with quantum entropy calculations on the 2-surface (for generic intersections of the curves with the surfaces, N is even).

该算符的优点是其谱等距: 本征值为 0 和 $\frac{\kappa\gamma}{2} N$ ，其中 $N \in \mathbb{N}$ 。这与二维曲面上的量子熵计算相容(对于曲线与曲面的一般相交情况，N 为偶数)。

The Angle Operator

角度算符

Having at our disposal the quantum triad-flux operator and the quantum area operator, we can define a quantum operator for the scalar product between the unit vectors normal to surfaces.

有了量子三元流算符和量子面积算符, 我们就可以为曲面单位法向量之间的标量积定义一个量子算符。

Consider two oriented surfaces $S, S' \subset \sum$ and a point $x \in S \cap S'$. We will start with constructing suitable classical expressions for unit normals at x . Let $S(\varepsilon) \subset S$ be a disc of coordinate radius ε centered at x . The limit

考虑两个定向曲面 $S, S' \subset \sum$ 和一个点 $x \in S \cap S'$ 。我们首先构造 x 处单位法向量的合适经典表达式。设 $S(\varepsilon) \subset S$ 是中心在 x 、坐标半径为 ε 的圆盘，该极限

$$n_{Sxi} := \lim_{\varepsilon \to 0} P_{S(\varepsilon),\tau_i} / A_{S(\varepsilon)} \tag{40}$$

provides the components of the unit co-vector $n_{Sxa} = n_{Sxi}e^i_a$ orthogonal to S and x . The scalar product with the other co-normal $n_{S'xi}e^i$ does not involve e_i anymore, because $q_{ij} = \delta_{ij}$, and $n_{Sxa}n_{S'xb}q^{ab} = n_{Sxi}n_{S'xj}q^{ij} = 1$.

给出了正交于 S 和 x 的单位余向量 $n_{Sxa} = n_{Sxi}e^i_a$ 的分量。与另一余法向量 $n_{S'xi}e^i$ 的标量积不再涉及 e_i ，因为 $q_{ij} = \delta_{ij}$ ，且 $n_{Sxa}n_{S'xb}q^{ab} = n_{Sxi}n_{S'xj}q^{ij} = 1$ 。

Now, turn to the quantum geometry, and consider the operator $\widehat{P}_{S(\varepsilon),\tau_i}/\widehat{A}_{S(\varepsilon)}$. The numerator and denominator commute, $\left[\widehat{P}_{S(\varepsilon),\tau_i}, \widehat{A}_{S(\varepsilon)}\right] = 0$, which makes the quotient independent of the ordering. The limit

现在转向量子几何，考虑算符 $\widehat{P}_{S(\varepsilon),\tau_i}/\widehat{A}_{S(\varepsilon)}$ 。分子与分母对易，$\left[\widehat{P}_{S(\varepsilon),\tau_i}, \widehat{A}_{S(\varepsilon)}\right] = 0$ ，因此商与排序无关。该极限

$$\hat{n}_{Sxi} = \lim_{\varepsilon \to 0} \frac{\widehat{P}_{S(\varepsilon),\tau_i}}{\widehat{A}_{S(\varepsilon)}} = \frac{\hat{J}_i^{x_0 S\uparrow} - \hat{J}_i^{x_0 S\downarrow}}{\sqrt{q^{ij}\left(\hat{J}_i^{x_0 S\uparrow} - \hat{J}_i^{x_0 S\downarrow}\right)\left(\hat{J}_j^{x_0, S\uparrow} - \hat{J}_j^{x_1, S\downarrow}\right)}} \tag{41}$$

is well defined on the states spanned by the eigenstates of the operator $\Delta_{S,x}$ of nonzero eigenvalues and satisfies, now on the quantum level, $q^{ij}\hat{n}_{Sxi}\hat{n}_{Sxj}q^{ij} = 1$. Finally, an operator giving the cosine of the angle between the surfaces S and S' at the point x may be defined by

在由非零本征值算符 $\Delta_{S,x}$ 的本征态张成的态上是良定义的，且在量子层面满足 $q^{ij}\hat{n}_{Sxi}\hat{n}_{Sxj}q^{ij} = 1$ 。最终，我们可以通过下式定义一个给出点 x 处曲面 S 和 S' 之间夹角余弦的算符

$$\cos\left(\hat{\alpha}_{SS'x}\right) := \frac{1}{2}q^{ij}\left(\hat{n}_{Sxi}\hat{n}_{S'xj} + \hat{n}_{S'xi}\hat{n}_{Sxj}\right). \tag{42}$$

For this operator to give a well-defined state, after the action on a spin-network state, it is necessary and sufficient that the graph has a nontrivial and generic intersection with each of the surfaces S and S' . Indeed, then 0 is not in the spectra of the area operators A_S and $A_{S'}$. This definition is inspired by [31]; however, it has been reformulated to work in the diffeomorphism covariant framework of the cylindrical function states, with the consequence that it has somewhat different properties.

要使该算符作用于自旋网络态后得到良定义的态，当且仅当图与每个曲面 S 和 S' 都存在非平凡的一般交点。此时 0 确实不在面积算符 A_S 和 $A_{S'}$ 的谱中。该定义受文献 [31] 启发；但我们对其进行重构，使之适用于柱函数态的微分同胚协变框架，因此该定义具备一些不同的性质。

The Volume Operators

体积算子

The 3-volume element defined by the variables (A, E) is $d^3x\sqrt{|\det E|}$. The integral of an arbitrary function F defined on $\sum$ can be approximated by using the fluxes through surfaces defined by a partition of $\sum$, with $\{x_n\} = S^a_n \cap S^b_n \cap S^c_n$, we have

由变量 (A,E) 定义的 3-体积元为 $d^3x\sqrt{|\det E|}$ 。定义在 $\sum$ 上的任意函数 F 的积分可以通过对 $\sum$ 分块得到的各曲面通量近似，结合 $\{x_n\} = S_n^a \cap S_n^b \cap S_n^c$ ，我们得到

$$\int_{\sum} d^3x\sqrt{|\det E|}F = \lim_{N\to\infty}\sum_n F(x_n)\sqrt{\left|\frac{1}{3!}\varepsilon^{ijk}\varepsilon_{abc}P_{S_n^a,\tau_i}P_{S_n^b,\tau_j}P_{S_n^c,\tau_k}\right|}. \tag{43}$$

Remarkably, the simple replacement of the classical fluxes by quantum operators again provides a well-defined finite operator in the Hilbert space $\mathcal{H}$ in the domain of the cylindrical functions [18, 28, 32, 33] . However, in this case, the result strongly depends on the choice of surfaces; in particular, it breaks the diffeomorphism invariance. Two different choices are pursued in the literature. They lead to two different diffeomorphism invariant quantum volume operators. In both cases, the regularizing 2-surfaces are adjusted to a given graph, used in the construction of a cylindrical function. In the first case, the internal regularization [32,33] and the triples of 2-surfaces intersect in the vertices of the graph (including spurious vertices in a refined graph). In the second case, the external regularization [28] and the 2-surfaces form cells that contain the vertices inside. We present both results below. In both regularizations, the quantum volume operator has a similar form with

值得注意的是，直接将经典通量替换为量子算子，即可在希尔伯特空间 $\mathcal{H}$ 的柱函数 [18, 28, 32, 33] 定义域内得到一个良定义的有限算子。但在这种情况下，结果强烈依赖于曲面的选取；尤其它会破坏微分同胚不变性。文献中采用了两种不同的选取方案，得到了两个不同的微分同胚不变量子体积算子。两种方案中，正则化 2-曲面都适配于构造柱函数所用的给定图。第一种方案是内部正则化 [32,33]，三组 2-曲面相交于图的顶点 (包括细化图中的伪顶点)。第二种方案是外部正则化 [28]，2-曲面构成单元将顶点包含在内部。我们在下文中给出两种方案的结果。两种正则化得到的量子体积算子形式相似，均为

$$\sqrt{\left|\widehat{\det E}\right|}(x) = (a_0\kappa\gamma)^{\frac{3}{2}}\sum_{y\in\sum}\delta^{(3)}(x,y)\sqrt{\hat{q}_y}, \tag{44}$$

here x runs through the entirety of $\sum$. The two regularizations differ in how they treat and define $\hat{q}_x$. Let us begin with the internal regularization. The diffeomorphism invariance can be ensured by a suitable averaging with respect to some family of choices, and the resulting $\hat{q}_x$ is given by

此处 x 遍历 $\sum$ 的全部元素。两种正则化的区别在于对 $\hat{q}_x$ 的处理和定义方式。我们先从内部正则化开始介绍。通过对一系列可选方案做适当平均即可保证微分同胚不变性，最终得到的 $\hat{q}_x$ 表达式为

$$\hat{q}_x^{\text{int}} = \left|\frac{1}{3!}\sum_{e,e',e''}\varepsilon^{ijk}\varepsilon(e,e',e'')\hat{J}_i^{x,e}\hat{J}_j^{x,e'}\hat{J}_k^{x,e''}\right|, \tag{45}$$

with $\varepsilon(e,e',e'') = \pm 1$ or 0 depending on the orientation of the tangent vectors at the point x , and the sum is over all triples of curves starting at x . Note that for cylindrical functions, only x , which is one of the vertices of a corresponding graph, and e, e', e'' that overlap edges of the graph at x contribute. The orientation-sensitive factor $\varepsilon(\cdot,\cdot,\cdot)$ annihilates all the planar triples, that is, planar vertices. In non-degenerate cases, it gives signs to the corresponding terms. The constant a_0 appearing in (44) is arbitrary and depends on the measure used for the averaging.

根据点 x 处切向量的取向，结果为 $\varepsilon(e,e',e'')=\pm1$ 或 0，求和覆盖所有从 x 出发的曲线三元组。注意对于柱函数，只有对应图的顶点 x ，以及在 x 处与图边重叠的 e,e',e'' 才有贡献。依赖取向的因子 $\varepsilon(\cdot,\cdot,\cdot)$ 会消去所有平面三元组，即平面顶点。在非退化情况下，它会为对应项赋予符号。(44) 中出现的常数 a_0 是任意的，依赖于平均所用的测度。

For the external regularization, one instead obtains

对于外部正则化，我们得到的结果是

$$\hat{q}_x^{\text{ext}} = \frac{1}{3!}\sum_{e\neq e'\neq e''\neq e}\left|\varepsilon^{ijk}\hat{J}_i^{x,e}\hat{J}_j^{x,e'}\hat{J}_k^{x,e''}\right|, \tag{46}$$

with the constant a_0 again arbitrary.

其中常数 a_0 同样是任意的。

Each of the operators, $\hat{q}_x^{\text{int}}$ and $\hat{q}_x^{\text{ext}}$, satisfies the commutation relations (35); hence they preserve the corresponding spin-network subspaces characterized above. And each of the operators is diffeomorphism covariant, in the sense that

算子 $\hat{q}_x^{\text{int}}$ 和 $\hat{q}_x^{\text{ext}}$ 都满足对易关系 (35)，因此它们都保持前文所述的对应自旋网子空间。并且两个算子都是微分同胚协变的，满足

$$U_\phi\hat{q}_xU_\phi^{-1} = \hat{q}_{\phi^{-1}(x)},\ U_\phi\hat{q}'_xU_\phi^{-1} = \hat{q}'_{\phi^{-1}(x)}.$$

There is a close relationship between the volume operator (44), with either regularization, and the quantization of volume based on discrete geometry [34], which we will turn to in section ”Quantum Tetrahedra”. In a nutshell, the volume spectrum one obtains from quantizing the phase space of Euclidean tetrahedra is very close to that of the operator (44) when acting on a 4-valent vertex. This convergence of different approaches is surprising and reassuring.

无论采用哪种正则化，(44) 给出的体积算子都与基于离散几何的体积量子化 [34] 密切相关，我们会在“量子四面体”一节讨论后者。简而言之，欧几里得四面体相空间量子化得到的体积谱，与算子 (44) 作用在 4 价顶点上得到的体积谱非常接近。不同方法得到的这一收敛结论既出人意料，也令人安心。

While these volume operators look similar, there is an essential qualitative difference between them. The operator $\hat{q}_x^{\text{ext}}$ depends on a number of different edges meeting at a vertex; however it is insensitive to the relationships between their tangent vectors: all the edges may be tangent to each other or contained in a single tangent plane. The operator $\hat{q}_x^{\text{int}}$ is useful in approaches that assume the existence of an underlying manifold and provide it with quantum geometry. Indeed, we expect the frame field determinant to distinguish planar from generic triples of vectors (In fact, the presence of the factor $\varepsilon(\cdot,\cdot,\cdot)$ complicates the analysis of $\hat{q}_x$ considerably, as it affects the spectrum, and for large valence, many sign configurations arise. For a 7-valent vertex, there are already $O\left(10^6\right)$ [35, 36] .). The second operator, $\hat{q}_x^{\text{ext}}$, on the other hand, is used in approaches where the manifold emerges together with geometry from a yet more fundamental discrete structure.

尽管这些体积算子看起来相似，它们之间存在本质的定性差异。算子 $\hat{q}_x^{\text{ext}}$ 依赖于相交于一个顶点的多条不同边，但它对这些边切向量之间的关系不敏感: 所有边可以彼此切向平行，或包含在同一个切平面内。算子 $\hat{q}_x^{\text{int}}$ 适用于假设基础流形预先存在并为其赋予量子几何的研究方案。实际上，我们可以预期标架场行列式能区分平面向量三元组和一般位置的向量三元组 (事实上，因子 $\varepsilon(\cdot,\cdot,\cdot)$ 的存在会大幅增加 $\hat{q}_x$ 的分析复杂度，因为它会影响谱，且对于高配价顶点会产生许多符号构型。对于 7 价顶点，已经存在 $O\left(10^6\right)$ [35, 36] 。) 另一方面，第二个算子 $\hat{q}_x^{\text{ext}}$ 则用于流形与几何一同从更基础的离散结构中涌现的研究方案。

The spectrum of the operator (44) is not known in general, but there are extensive partial analytic [34, 36-42] and numerical [35] results. We will touch on some of these below.

一般来说，(44) 中算子的谱尚未完全知晓，但已有大量部分解析结果 [34, 36-42] 和数值结果 [35]。我们会在下文中简要介绍其中部分内容。

Consistency checks on the quantization of volume have been proposed based on classical equalities involving the volume and fluxes (14) [39,40]. They concern both the averaging constant a_0 and the regularization scheme. In the case of the internal regularization, the results on the constant a_0 were ambiguous, namely, $a_0 = \frac{1}{2}$ or $a_0 = 1/2(2)^{\frac{1}{3}}$. The external regularization case was disfavored in some cases.

已有研究基于体积与流 (14) 之间的经典等式对体积量子化提出一致性检验 [39,40]，这些检验同时涉及平均常数 a_0 和正则化方案。在内部正则化情况下，关于常数 a_0 的结果是模糊的，即结果为 $a_0 = \frac{1}{2}$ 或 $a_0 = 1/2(2)^{\frac{1}{3}}$ 。部分研究中不支持外部正则化的情况。

We will consider the action of the operators in some special cases. The simplest nontrivial case is a 3-valent vertex, generic (i.e., non-planar) for $\hat{q}_x^{\text{int}}$ and arbitrary for the $\hat{q}_x^{\text{ext}}$. In that situation, both regularizations give rise to the same operator:

我们将考虑这些算子在若干特殊情形下的作用。最简单的非平凡情形是 3 价顶点: 对 $\hat{q}_x^{\text{int}}$ 而言是一般位置顶点 (即非平面顶点)，对 $\hat{q}_x^{\text{ext}}$ 而言是任意顶点。在此情形下，两种正则化得到的算子是相同的:

$$\varepsilon^{ijk}\hat{J}_i^e \otimes \hat{J}_j^{e'} \otimes \hat{J}_k^{e''} : (V_e \otimes V_{e'} \otimes V_{e''})_m^j \to (V_e \otimes V_{e'} \otimes V_{e''})_m^j \tag{47}$$

defined in each subspace of a triple of irreducible SU (2) representations $V_e \otimes V_{e'} \otimes V_{e''}$ characterized by a total spin j and an eigenvalue m of $\hat{J}_3^e + \hat{J}_3^{e'} + \hat{J}_3^{e''}$. In the case $j = 0$, the operator vanishes. That came as a surprise to those who discovered the volume operator of loop quantum gravity, who had wanted to restrict the theory to the 3-valent spin-network states (famous due to Penrose [43,44]), but in retrospect, it is clear that these states correspond to degenerate, planar geometries (cf. section ”Discrete Geometry”). (In fact, the result holds for an arbitrary n -valent vertex if the space of the invariant intertwiners, i.e., the space of states with $j = 0$, is one-dimensional [37].)

定义在由总自旋 j 和 $\hat{J}_3^e + \hat{J}_3^{e'} + \hat{J}_3^{e''}$ 的本征值 m 表征的不可约表示三元组 $V_e \otimes V_{e'} \otimes V_{e''}$ 的每个子空间中，其中每个子空间对应不可约 SU (2) 表示。在 $j = 0$ 的情况下，该算符为零。这对于圈量子引力体积算符的发现者来说十分意外，他们原本希望将理论限制在三价自旋网络态 (因彭罗斯 [43,44] 而闻名)，但回想后不难发现，这些态对应简并平面几何 (参见“离散几何”小节)。(事实上，若不变缠结算子空间即满足 $j = 0$ 的态空间是一维的，则该结果对任意 n 价顶点都成立 [37]。)

The spectrum for the three-valent case for non-gauge-invariant states is not known in closed form. But since it is important, we outline a few special cases here. The operator $\hat{q} = \left|\varepsilon^{ijk}\hat{J}_i \otimes \hat{J}_j \otimes \hat{J}_k\right|$, acts on the tensor product space $V_j \otimes V_{j'} \otimes V_{j''}$ and

非规范不变态的三价顶点谱尚无闭合形式。但由于该问题十分重要，我们在这里概述几个特殊情形。算子 $\hat{q} = \left|\varepsilon^{ijk}\hat{J}_i \otimes \hat{J}_j \otimes \hat{J}_k\right|$ 作用在张量积空间 $V_j \otimes V_{j'} \otimes V_{j''}$ 上，且

$$\mathrm{Tr}\left(\hat{q}^2\right) = \frac{2}{9} j\,(j+1)\, j'\,(j'+1)\, j''\,(j''+1)\,(2j+1)\,(2j'+1)\,(2j''+1)\,.$$

This gives an average eigenvalue squared $\left\langle \hat{q}^2 \right\rangle = \frac{2}{9} j\,(j+1)\, j'\,(j'+1)\, j''\,(j''+1)$. Special cases defined by $j_* = j_1 + j_2 + j_3 - 1$ give the eigenvalues q of $\hat{q}$ as

这给出了平均平方本征值 $\left\langle \hat{q}^2 \right\rangle = \frac{2}{9} j\,(j+1)\, j'\,(j'+1)\, j''\,(j''+1)$ 。由 $j_* = j_1 + j_2 + j_3 - 1$ 定义的特殊情形得到 $\hat{q}$ 的本征值 q 为

$$j = j_* \;\Rightarrow\; q = \pm\sqrt{j_1 j_2 j_3\,(j_* + 1)}, \tag{48}$$

$$j = j_* - 1 \Rightarrow q = 0, \pm\sqrt{j_*\,(4 j_1 j_2 j_3 - j_1 j_2 - j_2 j_3 - j_3 j_1) + j_1 j_2 j_3}\,. \tag{49}$$

Another important case is defined by a gauge-invariant 4-valent vertex v . In this case, $\hat{q}$ acts on $\mathrm{Inv}\big(V_j \otimes V_{j'} \otimes V_{j''} \otimes V_{j'}$, the subspace of $\left(V_j \otimes V_{j'} \otimes V_{j''} \otimes V_{j'''}\right)$ invariant under the diagonal action of SU (2) . In this case

另一个重要情形由规范不变的 4 价顶点 v 定义。在该情形下，$\hat{q}$ 作用于 $\mathrm{Inv}\left(V_j \otimes V_{j'} \otimes V_{j''} \otimes V_{j'''}\right)$ ，即 $\left(V_j \otimes V_{j'} \otimes V_{j''} \otimes V_{j'''}\right)$ 在 SU (2) 对角作用下不变的子空间。此时

$$\hat{q}_v^{\mathrm{ext}} = 4\left|\varepsilon^{ijk}\hat{J}_i^e \otimes \hat{J}_j^{e'} \otimes \hat{J}_k^{e''}\right| \quad \text{and} \quad \hat{q}_v^{\mathrm{int}} = \chi\,(e, e', e'', e''')\left|\varepsilon^{ijk}\hat{J}_i^e \otimes \hat{J}_j^{e'} \otimes \hat{J}_k^{e''}\right|,$$

(50)

where χ depends on a diffeomorphism class of the vertex and takes values $\chi\,(e, e', e'', e''') \in \{0, 1, 2, 3, 4\}$. For this case, the spectrum is non-degenerate, and there is a volume gap, i.e., a smallest nonzero volume eigenvalue [35, 42, 45] . The scaling of the smallest nonzero and largest volume eigenvalues has been derived in [42] and nicely corresponds with the classical geometry of tetrahedra. For much more discussion of these results, see section ”Discrete Geometry” and [34,42].

其中 χ 依赖于该顶点的微分同胚类，取值为 $\chi(e,e',e'',e''')\in\{0,1,2,3,4\}$。在该情形下，谱是非简并的，且存在体积隙，即一个最小非零体积本征值 [35,42,45]。最小非零体积本征值与最大体积本征值的标度关系已在文献 [42] 中推导得出，与四面体的经典几何完美契合。关于这些结果的更多讨论，参见“离散几何”一节以及文献 [34,42]。

For vertices of valence higher than 4, there are only numerical results for the volume spectrum. In [35], spectra of (44) for valences 5 to 7 and spins up to $j_{\max} = 13/2$ are calculated. Particular attention is paid to the lowest nonzero and the highest eigenvalue. It is found that the scaling behavior of the smallest nonzero eigenvalue with $j_{\max}$ depends crucially on the sign configuration coming from the embedding of the vertex and the factor $\varepsilon(\cdot,\cdot,\cdot)$. Generically, the volume gap increases with increase of $j_{\max}$. But there are sign configurations in which the volume gap shows the opposite behavior: in these cases, the lowest nonzero eigenvalue stays constant or decreases with increasing maximal spin. In fact, there are indications that the spectral density near zero may increase exponentially for odd valence, but the numerical data is not conclusive. The maximal eigenvalue increases with $j_{\max}$ for all sign configurations, but the rate depends on the sign configuration.

对于价数高于 4 的顶点，体积谱仅有数值结果。在文献 [35] 中，计算了价数 5 到 7、自旋不超过 $j_{\max} = 13/2$ 时式 (44) 的谱。研究重点放在最低非零本征值和最高本征值上。研究发现，最小非零本征值关于 $j_{\max}$ 的标度行为关键取决于顶点嵌入带来的符号构型与因子 $\varepsilon(\cdot,\cdot,\cdot)$。一般而言，体积隙随 $j_{\max}$ 增大而增加。但存在一些符号构型，体积隙表现出相反的行为: 在这些情形中，最低非零本征值随最大自旋的增大保持不变或减小。实际上，有迹象表明，对于奇数价，零点附近的谱密度可能呈指数增长，但数值数据尚无定论。对于所有符号构型，最大本征值都随 $j_{\max}$ 增大而增加，但其增长率依赖于符号构型。

The Inverse Metric Tensor Operator

逆度量张量算子

The inverse metric tensor in the connection frame variables is $q^{ab} = q^{ij}e_i^a e_j^b = E_i^a E_j^b / |\det E|$. To simplify q^{ab}, the idea is to probe it with a 1-form ω, construct a density of weight 1 that does not involve the inverse of E_i^a, and smear. The result is the observable

联络标架变量下的逆度量张量为 $q^{ab} = q^{ij}e_i^a e_j^b = E_i^a E_j^b / |\det E|$。为简化 q^{ab}，思路是用 1-形式 ω 对其进行探测，构造一个不包含 E_i^a 的逆、权重为 1 的密度，再做抹平处理，最终得到可观测量

$$\int_\Sigma d^3x \sqrt{\det q}\sqrt{q^{ab}\omega_a\omega_b} = \int_\Sigma d^3x \sqrt{E^a E^b \omega_a \omega_b}.$$

The corresponding operator

对应的算子

$$\int_\Sigma d^3x \sqrt{\widehat{E}^a \widehat{E}^b \omega_a \omega_b},\ \widehat{E}_i^a = -i\kappa\gamma\hbar\delta/\delta A_a^i,$$

applied to spin-network functions (27) and (33) turns out to be well defined, and the result is clear [46] - it acts as multiplication by the eigenvalue

作用于自旋网络函数 (27) 和 (33) 后是良定义的，结果十分清晰 [46]——它以乘上本征值的方式作用

$$\kappa\gamma\hbar\sum_{I=1}^{n}\sqrt{j_I\left(j_I+1\right)}\int_{p_I}\left|\omega_a\dot{p}_I^a\right|dt. \tag{51}$$

This operator was used in the quantization of the Hamiltonian of the Klein-Gordon field [47], and it may be applied in the case of vector fields too.

该算子已用于克莱因-戈登场哈密顿量的量子化 [47]，也可应用于矢量场的情况。

The Length Operator

长度算符

The orthogonal coframe e_a^i is expressed by the densitized frame E_i^a in a somewhat complicated way,

正交余标架 e_a^i 可通过密化标架 E_i^a 表示，形式相对复杂，

$$e_a^i = \frac{1}{2}\varepsilon_{abc}\varepsilon^{ijk}E_j^bE_k^c/\sqrt{|\det|\,E}, \tag{52}$$

which appears to complicate quantization. However, the discouraging denominator can be absorbed by the Poisson bracket with the volume observable [48, 49] ,

这会给量子化过程带来困难。不过，这个棘手的分母可以被体积可观测量 [48, 49] 的泊松括号吸收，

$$e_a^i = \frac{2}{\kappa\gamma}\left\{A_a^i, V\right\},\ V := \int_\Sigma d^3x\sqrt{|\det E|}. \tag{53}$$

Moreover, the length of a curve p can be expressed in terms of the holonomies, the volume, and the Poisson bracket [50],

此外，曲线 p 的长度可以用和乐、体积与泊松括号表示 [50]，

$$\int_p\sqrt{e^ie^jq_{ij}} = \kappa\gamma\lim_{N\to\infty}\sum_{n=1}^{N}\sqrt{-2\,\mathrm{Tr}\left(h_{\Delta p_n}^{-1}\left\{h_{\Delta p_n}, V\right\}h_{\Delta p_n}^{-1}\left\{h_{\Delta p_n}, V\right\}\right)}, \tag{54}$$

where $p = \Delta p_1 \circ \ldots \circ \Delta p_n$ is a partition uniformly refined as $n \to \infty$. After quantization, the quantum length operator becomes (one of the possibilities; [50])

其中 $p = \Delta p_1 \circ \ldots \circ \Delta p_n$ 是随 $n \to \infty$ 一致精细化的分划。量子化后，量子长度算符可表示为 (一种可能的形式，见 [50])

$$\int_p \sqrt{e^i e^j q_{ij}} = \frac{\kappa\gamma}{\hbar} \sum_{x\in\Sigma} \sqrt{-2\,\mathrm{Tr}\left(\left(h^{-1}_{\Delta p_x}\left[h_{\Delta p_x}, \hat{V}\right]\right)^{\dagger}\left(h^{-1}_{\Delta p_x}\left[h_{\Delta p_x}, \hat{V}\right]\right)\right)}. \tag{55}$$

When this operator is applied to a spin-network state, only a term corresponding to x at a vertex of the spin network contributes; this follows from the properties of the quantum volume operator.

当该算符作用于自旋网络态时，仅对应于 x 在自旋网络顶点处的项有贡献；这由量子体积算符的性质可得。

The Ricci Scalar Operator à la Regge

里奇标量算符: Regge 方法

The Riemann curvature tensor is defined by second derivatives of the triad E^a_i . There are currently no proposals for the quantization of such expressions that are diffeomorphism invariant on the space of cylindrical functions or the spin-network states. Remarkably, however, the integral of the Ricci scalar can be expressed as the limit of an expression that only uses the lengths of curves (hinges) and the deficit angles between surfaces [51]. These observables are available, as described in the previous sections. Specifically, consider a triangulation of Σ . It consists of three-dimensional tetrahedral cells, two-dimensional triangular faces, one-dimensional edges, and zero-dimensional vertices. Given a metric tensor q , we assign to every edge e in the triangulation the following data: (1) the length l_e and (2) the total angle θ_e given by the sum of the angles between the faces of the simplices that contain e , at a point $x \in e$. The following number converges to the integral of the scalar curvature of q_{ab} [51] when we refine the triangulation

黎曼曲率张量由三维标架 E^a_i 的二阶导数定义。目前，对于这类在柱函数空间或自旋网络态上满足微分同胚不变性的表达式，还没有任何量子化方案。但值得注意的是，里奇标量的积分可以表示为一个表达式的极限，该表达式仅用到曲线 (铰) 的长度以及曲面之间的亏缺角 [51]。正如前几节所述，这些可观测量是可构造的。具体来说，考虑 Σ 的一个三角剖分，它由三维四面体单元、二维三角形面、一维边和零维顶点组成。给定度量张量 q ，我们给三角剖分中的每条边 e 分配以下数据:(1) 长度 l_e ，(2) 在点 $x \in e$ 处，所有包含 e 的单形的面之间夹角之和得到的总角度 θ_e 。当我们细化三角剖分时，以下数值收敛到 q_{ab} 的标量曲率积分 [51]

$$\sum_e l_e\,(2\pi - \theta_e) \to \int_{\Sigma} d^3\sqrt{\det q}R. \tag{56}$$

As we know from the previous sections, the operator $\sum_e 2\pi\hat{l}_e - \frac{1}{2}\hat{l}_e\hat{\theta}_e - \frac{1}{2}\hat{\theta}_e\hat{l}_e$ is well defined in $\mathcal{H}$ and self-adjoint [52]. It may have a well-defined limit in some weak sense; the problem is its dependence on the original triangulation and series of refinements. These difficulties can be removed by an extra recipe of averaging with respect to the choices made [52].

正如我们从前几节所知，算符 $\sum_e 2\pi\hat{l}_e - \frac{1}{2}\hat{l}_e\hat{\theta}_e - \frac{1}{2}\hat{\theta}_e\hat{l}_e$ 在 $\mathcal{H}$ 中是良定义的自伴算符 [52]。它在某种弱意义下可以存在良定义极限；问题在于它依赖于初始三角剖分和一系列细化过程。这些困难可以通过额外对所选的剖分和细化做平均来解决 [52]。

Polyhedral States

多面体态

The picture of quantum states of geometry that emerges from loop gravity is that of curves forming a graph that contribute the flux of surface area and intersect in vertices from which quantized volumes arise. Even more specific geometric structure is brought by spin-network states constructed from so-called coherent intertwiners [53]. These intertwiners are defined as follows. Consider a vertex v of a graph whose edges are labelled by spins. Orient the edges to be outgoing, and number them $I = 1, \dots, n_v$. For every edge p_I , choose a normalized vector $u_I \in \mathfrak{su}(2)$ and a vector $|j_I, u_I\rangle$ in the corresponding representation $\mathcal{H}_{j_I}$ (also normalized) such that

圈引力得到的几何量子态图景是: 曲线构成图，曲线贡献表面积通量，曲线相交得到顶点，量子化体积由顶点产生。由所谓相干 intertwiners 构造的自旋网络态更进一步给出了具体几何结构 [53]。这些 intertwiners 定义如下: 考虑图的一个顶点 v ，其各边由自旋标记，将边定向为向外，编号为 $I = 1, \dots, n_v$ 。对每条边 p_I ，在对应表示 $\mathcal{H}_{j_I}$ 中选取一个单位向量 $u_I \in \mathfrak{su}(2)$ 和一个单位向量 $|j_I, u_I\rangle$ ，使得

$$u_I^i \hat{J}_i^{v,[p_I]} D_{(j_I)}{}^m{}_{m'}\left(h_{p_I}\right)\Big|j_I, u_I\Big\rangle^{m'} = \hbar j_I D_{(j_I)}{}^m{}_{m'}\left(h_{p_I}\right)\Big|j_I, u_I\Big\rangle^{m'}, \tag{57}$$

and the closure condition

且闭包条件

$$\sum_{I=1}^{n_v} j_I u_I = 0 \tag{58}$$

is satisfied. The corresponding coherent intertwiner $\iota\Big(j_1, u_1; \dots; j_{I_{n_v}}, u_{I_{n_v}}\Big)^{m_1 \dots m_{n_v}}$ is given by the average of $\otimes_{I=1}^{n_v} |j_I, u_I\rangle$ with respect to rotation action of $\mathrm{SU}(2)$. There exists a resolution of the identity in $\mathrm{Inv}\left(\otimes_{I=1}^{n_v} V_I\right)$ by the coherent intertwiners

成立。对应的相干 intertwiner $\iota\Big(j_1, u_1; \dots; j_{I_{n_v}}, u_{I_{n_v}}\Big)^{m_1 \dots m_{n_v}}$ 是 $\otimes_{I=1}^{n_v} |j_I, u_I\rangle$ 相对于 $\mathrm{SU}(2)$ 旋转作用的平均值。相干 intertwiners 在 $\mathrm{Inv}\left(\otimes_{I=1}^{n_v} V_I\right)$ 中存在单位分解

$$1^{m_1 \dots m_{n_v}}_{m_1' \dots m_{n_v}'} = \int d\mu\left(u_1, \dots, u_{n_v}\right) \iota^{m_1 \dots m_{n_v}} \iota^{\dagger}_{m_1' \dots m_{n_v}'} .$$

A geometric interpretation was provided in [45], where they found that the Minkowski theorem ensures existence of a polyhedron in $\mathbb{R}^3$ with facets orthogonal to the vectors u_I and facet areas j_I , provided the closure condition is satisfied. In this manner, a quantum polyhedron can be associated with each vertex of a spin network. The polyhedra assigned to different vertices are related by the edges connecting the vertices and consequently share face areas. However, the connected faces generically have different shapes, e.g., a triangular face glued to a quadrilateral one. This geometric description is often referred to as a twisted geometry [54]. Remarkably, this brings together the spin network states of loop quantum gravity and the quantization of discrete geometry as described in section "Discrete Geometry" of this chapter.

文献 [45] 给出了几何解释，研究发现，只要满足闭包条件，闵可夫斯基定理就能保证在 $\mathbb{R}^3$ 中存在一个多面体，其面正交于向量 u_I ，面面积为 j_I 。通过这种方式，每个自旋网络顶点都可以关联一个量子多面体。分配给不同顶点的多面体通过连接顶点的边关联，因此共享面面积。但连接的面通常形状不同，例如三角形面粘接在四边形面上。这种几何描述通常被称为扭曲几何 [54]。值得注意的是，该描述将圈量子引力的自旋网络态，与本章“离散几何”一节介绍的离散几何量子化结合在了一起。

Coherent States

相干态

The states discussed in the previous section minimize and distribute the uncertainties in the non-commuting P -variables (30) coherently. Another interesting task, especially if the quantization of four-dimensional spacetime geometries is the ultimate goal, is to balance uncertainties between the P -variables and the holonomies. While the spin-network states minimize the fluctuations of the densitized triads and leave the fluctuations of the holonomies maximal, it is desirable to find different states that balance these uncertainties. Such states are also loosely called semiclassical states.

上一节讨论的态对非对易 P 变量 (30) 的不确定性进行了相干的最小化与分配。若四维时空几何的量子化是最终目标，那么还有一项尤为有意思的任务: 平衡 P 变量与和乐之间的不确定性。自旋网络态最小化了致密化三分量 triad 的涨落，却让和乐的涨落达到最大，因此我们亟需找到能平衡这些不确定性的不同态。这类态也被宽泛地称为半经典态。

A proposal for these states that is well developed is that of group coherent states [15, 55-60]. It is based on coherent states on groups [61, 62]. The basic idea is not to consider spin network states but infinite coherent linear combinations of such states, which increase the uncertainties of the P -variables and reduce those of the holonomies. The main technical tool is to generalize standard coherent states to ones on the quantization of phase spaces of the form T^*G for G a compact abelian group, in particular for $T^*\mathrm{SU}(2)$. A coherent state on $\mathrm{SU}(2)$ is an element of $L^2(\mathrm{SU}(2))$ and takes the form

目前已发展成熟的这类态方案是群相干态 [15, 55-60]，它建立在群上相干态的基础上 [61, 62]。其基本思路是不单独考虑自旋网络态，而是构造这类态的无穷相干线性组合，以此增大 P 变量的不确定性、减小和乐的不确定性。该方案的核心技术手段是将标准相干态推广到形如 T^*G 的相空间量子化上，其中 G 为紧阿贝尔群，具体而言是 $T^*\mathrm{SU}(2)$ 。$\mathrm{SU}(2)$ 上的相干态是 $L^2(\mathrm{SU}(2))$ 中的一个元素，形式为

$$\Psi_k^t(g) = \lim_{h\to k} e^{-t\Delta_{\mathrm{SU}(2)}}\delta\left(gh^{-1}\right),\ k \in \mathrm{SL}(2,\mathbb{C}), \tag{59}$$

where $\Delta_{\mathrm{SU}(2)}$ is the Laplacian on $\mathrm{SU}(2)$, δ is the group delta function peaked on the unit element, k is an element of the complexification $\mathrm{SU}(2)_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$, and t is a positive real parameter. An important property of the operator that acts on the Dirac delta on the right-hand side is that it maps it into an analytic function on SU(2). Thanks to analyticity, the function extends to $\mathrm{SL}(2,\mathbb{C})$ and the variable gh^{-1} . One can show [15, 55, 56] that $\Psi_k^t(g)$ is peaked on a certain element (h,t) of $T^*\mathrm{SU}(2)$, in the sense that $|\Psi|^2$ is peaked on h and the expectation values of functions f of invariant vector fields are close to $f(t)$, with small fluctuations.

其中 $\Delta_{\mathrm{SU}(2)}$ 是 SU (2) , δ 上的拉普拉斯算子，是集中在单位元上的群 δ 函数，k 是复化 $\mathrm{SU}(2)_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ 中的一个元素，t 是正实参数。右侧作用在狄拉克 δ 函数上的算子有一个重要性质: 它会将狄拉克 δ 函数映射为 SU(2) 上的解析函数。得益于解析性，该函数可延拓至 SL (2, ℂ) 和变量 gh^{-1} 。可以证明 [15, 55, 56] ，$\Psi_k^t(g)$ 集中在 $T^*\mathrm{SU}(2)$ 的某个特定元素 (h,t) 上: 具体来说，$|\Psi|^2$ 集中在 h ，不变向量场函数 f 的期望值接近 $f(t)$ ，且涨落很小。

Moreover, the parameter t balances fluctuations between those of multiplication operators and those of the invariant derivatives in $L^2(\mathrm{SU}(2))$.

此外，参数 t 平衡了 $L^2(\mathrm{SU}(2))$ 中乘法算子涨落与不变导数涨落之间的平衡。

From these coherent states in $L^2(\mathrm{SU}(2))$, coherent states for loop quantum gravity can be constructed as product states

从 $L^2(\mathrm{SU}(2))$ 中的这些相干态出发，可以构造出圈量子引力的相干态，其形式为乘积态

$$\Psi_\gamma = \prod_{p\in\gamma} \Psi_{k_p}^{t_p}\left(h_p\right) \tag{60}$$

where the parameters k_p, t_p are chosen judiciously and adapted to the classical intrinsic and extrinsic geometry that is to be approximated [15, 55-58]. To enforce gauge invariance, these states have to be suitably adjusted [59,63].

其中参数 k_p, t_p 经过精心选取，适配待近似的经典内蕴与外蕴几何 [15, 55-58]。为了保证规范不变性，这些态还需要经过适当调整 [59,63]。

These ideas can also be extended to graphs with infinitely many edges [64,65]. This construction already leaves the Ashtekar-Lewandowski Hilbert space. Further generalizations of the same idea are general complexifier coherent states [66] and r-Fock measures, which we will discuss in section ”Other Phases of Quantum Geometry”.

这些思路也可以推广到有无穷多条边的图上 [64,65]。该构造已经超出了阿西卡-莱万多夫斯基希尔伯特空间的范围。同一思路的进一步推广得到了通用复化器相干态 [66] 与 r-福克测度，我们会在“量子几何的其他相”一节讨论这些内容。

Quantum-Reduced Gravity

约化量子引力

There is a class of spin-network states, and we call them reduced, for which the action of the flux operators, as well as all the others expressed in terms of the fluxes, is greatly simplified. The associated approximation is that the spins carried by the edges are large $j_I \gg 1$, and we consider the leading order in j ’ s only. The word ”reduced” comes from a class of models of LQG in which one fixes coordinates in $\sum$ and restricts the space of data (A, E) so that the resulting metric tensor is diagonal and E is suitably rotated at each point (see, e.g.,[67,68]). The states of those models can be identified as states of full LQG, of the type mentioned above, and described more precisely below, with the caveat that they are not gauge-invariant [69].

存在一类被我们称为约化自旋网络态，通量算符以及所有其他可由通量表示的算符对这类态的作用都被大大简化。对应的近似条件是：边携带的自旋很大 $j_I \gg 1$，并且我们仅保留 j 中的领头阶。“约化”一词来源于圈量子引力中的一类模型，这类模型中，我们在 $\sum$ 中固定坐标，并约束数据空间 (A, E)，使得最终得到的度量张量是对角的，且 E 在每一点都经过了适当旋转 (参见例如文献 [67,68])。这些模型的态可以对应为上文提到、下文将更精确描述的完整圈量子引力的态，但需要注意，这类态并不满足规范不变性 [69]。

A reduced spin-network is defined on a (topologically) cubic graph in $\sum$. Taking advantage of that topology, the edges of the graph can be grouped into three classes corresponding to the would be (if $\sum$ were endowed with a flat geometry and the graph would be geometrically cubic) three directions, say $i = x, y, z$, and oriented accordingly. A one-to-one correspondence is introduced between the directions of the edges and orthonormal generators $\tau_i = \tau_x, \tau_y, \tau_z \in \mathfrak{su}(2)$. In the irreducible representation V_{p_I}, we choose an orthonormal basis of the eigenvectors of the spin operator $D'_{(j_I)}(\tau_{i_I})$, where $i_I = x, y$ or z is the direction of p_I. We label the elements of the basis by the eigenvalues $m = -j_I, \dots, j_I$. Given that data, to every edge p_I, there is assigned an entry, either $D_{(j_I)}{}^{j_I}_{j_I}$ or $D_{(j_I)}{}^{-j_I}_{-j_I}$. The corresponding cylindrical function is

约化自旋网络定义在 $\sum$ 中的 (拓扑) 立方图上。利用该拓扑性质，图的边可以分为三类，分别对应三个方向 (若 $\sum$ 赋予平坦几何且该图是几何立方，这三个方向就是实际的空间方向)，记为 $i = x, y, z$，并按方向定向。我们在边方向和正交归一生成元 $\tau_i = \tau_x, \tau_y, \tau_z \in \mathfrak{su}(2)$ 之间建立了一一对应关系。在不可约表示 V_{p_I} 中，我们为自旋算符 $D'_{(j_I)}(\tau_{i_I})$ 的本征向量选取正交归一基，其中 $i_I = x, y$ 或 z 是 p_I 的方向。我们用本征值 $m = -j_I, \dots, j_I$ 标记基矢。给定这些数据后，每条边 p_I 都会被赋予一个输入，即 $D_{(j_I)}{}^{j_I}_{j_I}$ 或 $D_{(j_I)}{}^{-j_I}_{-j_I}$。对应的柱函数为：

$$\Psi(A) = \prod_I \sqrt{j_I(j_I+1)} D_{(j_I)}{}^{\pm J_I}_{\pm J_I}\left(h_{p_I}(A)\right), \tag{61}$$

where either ++ or – is chosen independently on each edge p_I of the graph and in an arbitrary way depending on the state. Notice that the reduced spin-networks are not gauge invariant though.

其中，每条边 p_I 都可以独立选择 ++ 或–，选择方式任意，由态决定。但请注意，约化自旋网络并不满足规范不变性。

All the operators of the geometry introduced in this chapter were written in terms of the vertex-edge spin operators (29). Their action on the reduced states is very simple. Indeed, if v is a vertex of a given reduced spin-network state and p_I is an edge that meets v and has direction x, then

本章介绍的所有几何算符都可以用顶点-边自旋算符 (29) 表示。它们在约化态上的作用非常简单：若 v 是某个给定约化自旋网络态的顶点，p_I 是一条连接 v、方向为 x 的边，则

$$\hat{J}_i^{v[p_I]}\Psi = \begin{cases} \frac{\pm j_I \hbar}{2}\Psi, & \text{if } i = x \text{ and } p_I \text{ is outgoing} \\ \frac{\mp j_I \hbar}{2}\Psi, & \text{if } i = x \text{ and } p_I \text{ is incoming} \\ O\left(\sqrt{j_I}\right) & \text{if } i = y, z. \end{cases} \tag{62}$$

Owing to these simplifications, the contribution from a vertex v to the quantum volume element (44) is

由于这些简化，顶点 v 对量子体积元 (44) 的贡献为

$$\left(\frac{\kappa\gamma\hbar}{2}\right)^{\frac{3}{2}}\sqrt{\left(j_x^- \pm j_x^+\right)\left(j_y^- \pm j_y^+\right)\left(j_z^- \pm j_z^+\right)} + O\left(\sqrt{j}\right), \tag{63}$$

where $j_i^{\pm}$ is the spin carried by an outgoing/incoming edge, and the spins add if both magnetic numbers have the same sign. Incidentally, according to the reduced states, the unknown factor a_0 should be chosen to be $\frac{1}{2}$ as one of the distinguished possibilities mentioned above.

其中 $j_i^{\pm}$ 是出射/入射边携带的自旋，若两个磁数符号相同，则自旋相加。另外，根据约化态的性质，未知因子 a_0 应当取为 $\frac{1}{2}$，这是前文提到的特殊可能取值之一。

The subspace of reduced states is preserved by the action of the holonomy operators of spins $s \ll j$ (up to lower order terms in j). Specifically,

约化态的子空间在自旋 $s \ll j$ 的全纯算符作用下保持不变 (差一个 j 的低阶项)，具体来说:

$$D_{(j_J)^m_n}\left(h_{p_J}(A)\right)\Psi(A) = \prod_{I\neq J}\sqrt{j_I\left(j_I+1\right)}D_{(j_I)_\pm}j_{J_I}\left(h_{p_I}(A)\right)$$

$$\begin{cases} \sqrt{(j_J+m)(j_J+m+1)}D_{(j_J+m)\pm J_I+m}\left(h_{p_J}(A)\right) & \text{if } n=m \\ O\left(\frac{1}{\sqrt{j}}\right) & \text{if } n\neq m \end{cases} \tag{64}$$

For the subspace of states spanned by cubic spin networks, there is an alternative to the abovementioned quantization of the curvature scalar. In particular, the action of the curvature scalar operator has been computed on the reduced states in [70, 71] .

对于立方自旋网络张成的态空间，上文提到的曲率标量量子化之外还有另一种方案。具体而言，已有研究在 [70, 71] 中计算了曲率标量算符在约化态上的作用。

Other Phases of Quantum Geometry

量子几何的其他相

In this section, we will consider some extensions of the quantized geometry that we have described in the previous sections. We will proceed from straightforward extensions of the formalism to phases of quantum geometry that are less directly related. The properties of some associated ground states are sketched in Fig. 1.

本节中，我们将讨论前文所述量子化几何的若干扩展。我们将从形式体系的直接推广逐步过渡到关联更间接的量子几何相，部分相关基态的性质见图 1 的概览。

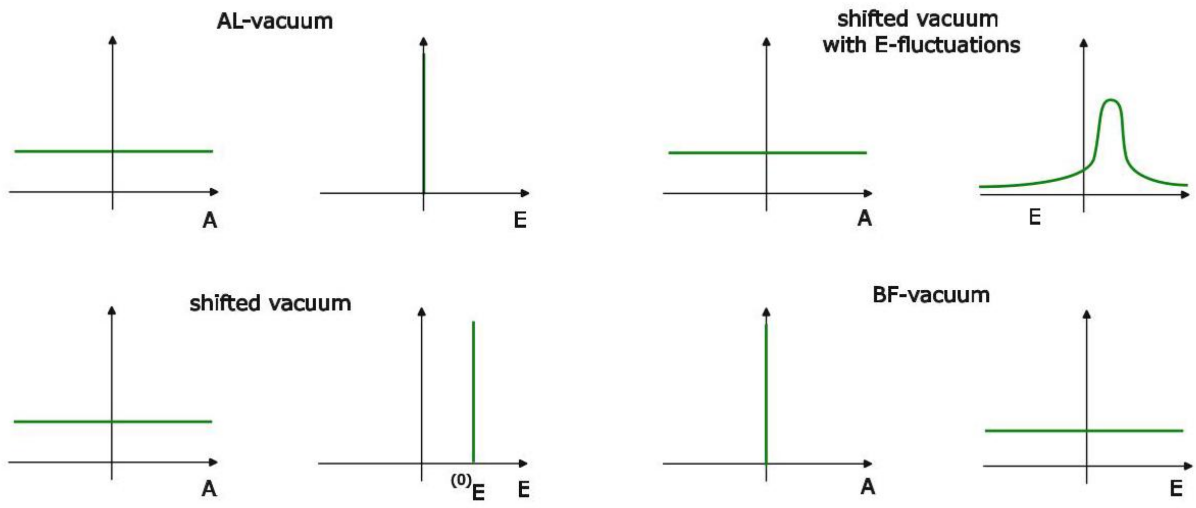


Fig. 1 Schematic depiction of the properties of various ground states of quantum geometry. Sketched is the behavior in the A - and E -representation

图 1 不同量子几何基态的性质示意图。图中展示了该性质在 A 表象和 E 表象中的行为

The first extension is the introduction of matter. Scalar fields, gauge fields, and fermionic fields can be coupled to gravity quantized as described in section ”Ashtekar-Lewandowski Representation”. The resulting Hilbert space is a subset of the tensor product

第一个扩展是引入物质。标量场、规范场和费米场都可以与按照“阿西特卡-莱万多夫斯基表象”一节所述方法量子化的引力耦合，得到的希尔伯特空间是张量积的一个子集

$$\mathcal{H} = \mathcal{H}_{\mathrm{AL}} \otimes \mathcal{H}_{\mathrm{matter}} \tag{65}$$

Details depend on the kind of matter that is coupled: Gauge fields with a compact structure group G can also be quantized in terms of holonomies and fluxes, so effectively the only change is in the structure group, which becomes a tensor product $\mathrm{SU}(2) \times G$ [72]. This does not change any of the features of the quantum geometry described in section ”Quantum Geometry”. The same can be said of scalar fields [73], which are quantized in analogy with gravity as described in section ”Ashtekar-Lewandowski Representation”.

细节取决于耦合的物质种类: 结构群为紧致群 G 的规范场同样可以用全同伦和通量量子化，因此实际上唯一的改变是结构群变为张量积 $\mathrm{SU}(2) \times G$ [72]。这不会改变“量子几何”一节所述量子几何的任何特征。标量场也是如此 [73]，它按照“阿西特卡-莱万多夫斯基表象”一节所述方法，类比引力完成量子化。

A first change of the properties of the quantum geometry happens with the inclusion of fermions. This is because the fermion fields transform under the same gauge group as gravity. In loop quantum gravity, the fermions naturally appear as point-like excitations [73-76]. To obtain gauge-invariant states, excitations of gravity and fermions have to be coupled. For the quantum geometry, this means that the gravitational part can transform nontrivially under gauge transformations. Intertwiners at a vertex in the presence of a fermion must now also couple to the fermion. In the case of a Weyl fermion, this means that an intertwiner ι lives in the space $\iota \in \mathrm{Inv}\left(j_1 \ldots, j_n, \frac{1}{2}\right)$. As explained in section "The Angle Operator", this changes the volume spectrum, sometimes dramatically. The volume operator is now nontrivial on 3-valent vertices that carry a fermionic excitation. For the intertwiner $\iota \in \mathrm{Inv}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, one obtains $|q|\, \iota = \sqrt{3}\iota/4$, via (48), for example.

纳入费米子后, 量子几何的性质才首次发生改变。这是因为费米场与引力在同一个规范群下变换。在圈量子引力中, 费米子自然表现为点状激发 [73-76]。要得到规范不变态, 引力激发和费米子激发必须耦合。对量子几何而言,这意味着引力部分在规范变换下可以发生非平凡变换。存在费米子时, 顶点处的交缠子必须额外与费米子耦合。对于外尔费米子, 这意味着交缠子 ι 属于空间 $\iota \in \mathrm{Inv}\left(j_1 \ldots, j_n, \frac{1}{2}\right)$ 。正如“角度算符”一节所述, 这会改变体积谱, 有时改变非常显著。携带费米激发的三价顶点上, 体积算符现在是非平凡的。例如, 对于交缠子 $\iota \in \mathrm{Inv}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, 通过式 (48) 可以得到 $|q|\, \iota = \sqrt{3}\iota/4$ 。

Matter also allows the construction of new geometric operators that relate matter and geometry (see, e.g., [47] for a scalar field and [77] for fermions).

物质还允许我们构造关联物质与几何的新型几何算符 (例如, 标量场相关见 [47], 费米子相关见 [77])。

In the supersymmetric extension of loop quantum gravity [78-84], further types of matter have to be considered and quantized [80-82]. The supergroup OSp $(\mathcal{N} \mid 2)$ replaces the direct product of (65) [83,84]. The gauge transformations thus mix matter and gravity degrees of freedom. In a formulation partially preserving manifest supersymmetry, fermionic excitations cannot therefore be point-like but also extend along edges in $\sum$. Details of these constructions can be found in ▹ Chap. 85, "Hamiltonian Theory: Generalizations to Higher Dimensions, Supersymmetry, and Modified Gravity" of this volume.

在圈量子引力的超对称推广中 [78-84], 必须考虑并量子化更多种类的物质 [80-82]。超群 OSp $(\mathcal{N} \mid 2)$ 取代了 (65) 的直积 [83,84]。因此规范变换会混合物质与引力的自由度。在一个部分保留显式超对称性的表述中, 费米激发因此不能是点状的, 还会沿 $\sum$ 中的边延展。这些构造的细节可以在本卷 ▹ 第 85 章“哈密顿理论: 向高维、超对称与修正引力的推广”中找到。

An extension of the formalism of section "Ashtekar-Lewandowski Representation" comes in the form of different representations of the holonomy-flux algebra $\mathfrak{A}_{\mathrm{HF}}$ described in section "The Quantum Holonomy-Flux Variables". The Ashtekar-Lewandowski representation is fundamental to loop quantum gravity because it is the only possible representation of $\mathfrak{A}_{\mathrm{HF}}$ that carries a unitary representation of spatial diffeomorphisms and has a diffeomorphism invariant cyclic vector. One can compare the situation to that of relativistic quantum field theory. For free fields, with specified mass and spin or helicity, there is only one positive energy representation that carries a unitary representation of the Poincare group and contains a cyclic Poincare-invariant vector: the vacuum. This representation is certainly fundamental, but there are other representations that are interesting and relevant, such as those for QFT at a finite temperature. The same can be said for loop quantum gravity. As already pointed out in section "The Quantum Holonomy-Flux Variables", there are several definitions of $\mathfrak{A}_{\mathrm{HF}}$ that differ regarding the imposition of certain higher-order commutation relations. This also affects the representation theory. Details can be found in [16, 17, 85, 86] . The following applies to the definition given in [17].

“阿西特卡-莱万多夫斯基表示”一节形式体系的一种推广，是对“量子全纯-通量变量”一节所述的全纯-通量代数 $\mathfrak{A}_{\mathrm{HF}}$ 采用不同表示形式。阿西特卡-莱万多夫斯基表示对圈量子引力而言具有基础性，因为它是 $\mathfrak{A}_{\mathrm{HF}}$ 唯一同时满足拥有空间微分同胚的幺正表示、且存在微分同胚不变循环向量的表示。这一情况可以和相对论量子场论做类比。对于给定质量、自旋或螺旋度的自由场，只有一种正能表示同时满足拥有庞加莱群的幺正表示、且包含循环庞加莱不变向量——即真空。这种表示当然是基础性的，但也存在其他有趣且相关的表示，例如有限温度量子场论的表示。圈量子引力也是如此。正如“量子全纯-通量变量”一节已经指出的，$\mathfrak{A}_{\mathrm{HF}}$ 存在多种定义，这些定义对某些高阶对易关系的施加要求不同，这也会影响表示理论，详见 [16, 17, 85, 86] 。以下内容针对文献 [17] 给出的定义展开。

Given a classical background field ${}^{(0)}E$, one can define [17, 87, 88] ,

给定经典背景场 ${}^{(0)}E$ ，可以定义 [17, 87, 88] ,

$$\pi\left(P_{S,f}\right)=\widehat{P}_{S,f}+P_{S,f}\left({}^{(0)}E\right)\mathbb{1}, \tag{66}$$

where $\widehat{P}$ is the operator (30) in the Ashtekar-Lewandowski representation. The Hilbert space and the action of the holonomies are unchanged. This representation describes quantum excitations over a background geometry given by ${}^{(0)}E$. This becomes even more clear when considering geometric operators that can be defined by a similar regularization procedure as in the Ashtekar-Lewandowski representation [88]. The volume operator for a spatial region R , for example, becomes $\widehat{V}(R) = \widehat{V}_{\mathrm{AL}}(R)+{}^{(0)}V(R)$, where $\widehat{V}_{\mathrm{AL}}$ is the volume operator in the Ashtekar-Lewandowski representation and ${}^{(0)}V(R)$ is the volume of R in the background geometry.

其中 $\widehat{P}$ 是阿西特卡-莱万多夫斯基表示中的算符 (30)，希尔伯特空间与全纯的作用保持不变。该表示描述了背景几何 ${}^{(0)}E$ 上的量子激发，当考虑可以通过阿西特卡-莱万多夫斯基表示中类似的正则化过程定义的几何算符时，这一点会更加清晰 [88]。例如，空间区域 R 的体积算符变为 $\widehat{V}(R) = \widehat{V}_{\mathrm{AL}}(R)+{}^{(0)}V(R)$ ，其中 $\widehat{V}_{\mathrm{AL}}$ 是阿西特卡-莱万多夫斯基表示中的体积算符，${}^{(0)}V(R)$ 是背景几何中 R 的体积。

It is also possible to include nonzero background fluctuations, by including a suitable operator valued shift, $\pi\left(P_{S,f}\right)=\widehat{P}_{S,f}+{}^{(0)}\widehat{P}_{S,f}$, which can, for example, induce Gaussian fluctuations of P in the vacuum state [89]. In these constructions, the background geometry is fixed, and it cannot be changed by the elements of $\mathfrak{A}_{\mathrm{HF}}$. This can be modified [85, 86, 90] by suitably enlarging $\mathfrak{A}_{\mathrm{HF}}$ by elements of the form

我们也可以通过引入合适的算符值位移来包含非零背景涨落，即 $\pi\left(P_{S,f}\right)=\widehat{P}_{S,f}+{}^{(0)}\widehat{P}_{S,f}$ ，例如它可以在真空态中诱导出 P 的高斯涨落 [89]。在这类构造中，背景几何是固定的，无法被 $\mathfrak{A}_{\mathrm{HF}}$ 的元素改变。这一点可以修改: 通过用如下形式的元素适当扩大 $\mathfrak{A}_{\mathrm{HF}}$ ，得到 [85, 86, 90]

$$\widehat{W}\left({}^{(0)}E\right)=\exp\int {}^{(0)}E\cdot\widehat{A} \tag{67}$$

(see [16] for subtleties in the case of a non-abelian structure group). Considering representations, one finds that these elements can shift the background. (For a schematic depiction of the properties of these representations, see Fig. 1.)

(非阿贝尔结构群情形的微妙之处参见 [16])。考虑表示时可以发现，这些元素能够移动背景。(这些表示的性质示意图参见图 1。)

If one is willing to go away from the algebra $\mathfrak{A}_{\mathrm{HF}}$ and to introduce additional structures, other constructions become possible. One possibility is a quantum theory based on a vacuum that has, in some sense, properties opposite to those of the Ashtekar-Lewandowski vacuum. The BF-vacuum [91-95] is an eigenstate of holonomies with the eigenvalues of a flat connection ${}^{(0)}A$, that is, $h|0\rangle_{\mathrm{BF}} = {}^{(0)}h|0\rangle_{\mathrm{BF}}$. The observables related to E have maximum uncertainties and act by creating singular, two-dimensional excitations over the flat connection ${}^{(0)}A$. The construction of this new vacuum necessitates the introduction of a new algebraic structure comprising holonomies and fluxes and based on a class of two-complexes and their duals. However, it has been shown [95] that for the case of structure group U(1), the analog to the BF vacuum can be obtained in a representation of a continuum theory, without any discretization. In that theory, the discreteness emerges only on the quantum level as a property of the spectrum of the quantum holonomy operators.

如果我们愿意脱离代数 $\mathfrak{A}_{\mathrm{HF}}$ 并引入额外结构，其他构造就成为可能。一种可能性是基于真空的量子理论，该真空在某种意义上具有与阿西特卡-莱万多夫斯基真空完全相反的性质。BF 真空 [91-95] 是和乐的本征态，对应平联络 ${}^{(0)}A$ 的本征值，即 $h|0\rangle_{\mathrm{BF}} = {}^{(0)}h|0\rangle_{\mathrm{BF}}$。与 E 相关的可观测量具有最大不确定性，并且会在平联络 ${}^{(0)}A$ 上通过作用产生奇异的二维激发。构造这种新真空需要引入包含和乐与通量的新代数结构，该结构基于一类二维复形及其对偶。然而已有研究表明 [95]，对于结构群 U(1) 的情况，BF 真空的类比可以在连续统理论的表示中得到，无需任何离散化。在该理论中，离散性仅在量子层面作为量子和乐算子谱的性质涌现出来。

In a similar spirit, one can keep the algebra generated by the holonomy functionals untouched but change the rest of the relations in $\mathfrak{A}_{\mathrm{HF}}$ by introducing a more regular smearing of the fields E (three additional integrations against a smooth kernel). One obtains a new algebra [96, 97], which remarkably is, in the Abelian case where $\mathrm{SU}(2)$ is replaced by $\mathrm{U}(1)$, isomorphic to the algebra underlying the Fock representation of the quantum electromagnetic field. It can be used to define new representations of the holonomy part of the $\mathrm{U}(1)$ -analog of $\mathfrak{A}_{\mathrm{HF}}$ in which the holonomies have Gaussian fluctuations, the r-Fock representations. Closely related are constructions of new states with Gaussian fluctuations, the complexifier coherent states [15,55,56,66]. It is possible to extend some of these constructions to the case of linearized gravity [98], scalar fields [99], and even non-Abelian gauge fields [100]. Also, shadow states in the Ashterkar-Lewandowski Hilbert space can be defined using the r-Fock representations [101], which retain some of the properties of these representations.

类似地，我们可以保留和乐泛函生成的代数不变，而是通过对场 E 引入更正则的涂抹 (对光滑核额外进行三次积分) 来修改 $\mathfrak{A}_{\mathrm{HF}}$ 中的其余关系，由此得到一个新代数 [96, 97]，值得注意的是，在将 $\mathrm{SU}(2)$ 替换为 $\mathrm{U}(1)$ 的阿贝尔情形下，这个新代数与量子电磁场福克表示基础的代数同构。它可以用来定义 $\mathfrak{A}_{\mathrm{HF}}$ 的 $\mathrm{U}(1)$ 类比中，和乐部分的新表示，其中和乐具有高斯涨落，即 r-福克表示。与之密切相关的是具有高斯涨落的新态构造，即复相干态 [15,55,56,66]。其中部分构造可以拓展到线性引力 [98]、标量场 [99] 甚至非阿贝尔规范场 [100] 的情况。此外，还可以利用 r-福克表示在阿西特卡-莱万多夫斯基希尔伯特空间中定义影态 [101]，这类影态保留了这些表示的部分性质。

A very interesting generalization of the formalism of loop quantum gravity is to quantum group valued connections [102]. In this case, the algebra of cylindrical functions over the group SU(2) is replaced by a (non-commutative) algebra of functions over a compact quantum group. It turns out that the co-multiplication of the quantum group, together with a certain quantum group automorphism, is precisely the structure needed to define an algebra of cylindrical functions as an inductive limit in this case [102].

圈量子引力形式非常有意思的一个推广是量子群值联络 [102]。在这种情况下，SU(2) 群上的柱函数代数被替换为紧量子群上的 (非交换) 函数代数。可以证明，量子群的余乘，结合特定的量子群自同构，恰好就是在这种情况下，将柱函数代数定义为归纳极限所需的结构 [102]。

Another generalization is to equip the spin network states of geometry with further structure that allows to encode the topology of the spatial slice $\sum$ [103-105]. Using this topspin-formalism, it might be possible to describe topology change and quantum superposition of topologies.

另一种推广是给几何的自旋网态添加额外结构，用以编码空间切片 $\sum$ 的拓扑 [103-105]。利用这种顶自旋形式，有可能描述拓扑变化和拓扑的量子叠加。

A link between non-commutative geometry and loop quantum gravity was established by introducing a semi-finite spectral triple over the space of connections [106]. The triple involves an algebra of holonomy loops and a Dirac type operator, which resembles a global functional derivation operator.

非交换几何与圈量子引力之间的联系，是通过在联络空间上引入半有限谱三元组建立的 [106]。该三元组包含一个和乐圈代数和一个狄拉克型算子，后者类似全局泛函导数算子。

Another possibility to extend the formalism while changing some technical aspects is group field theory (GFT) (There is by now an extensive literature on various aspects of this theory. For an introduction with a view toward loop quantum gravity, see [107].). One possible application is as an analog of many-particle theory in geometry; here, the analog of the one-particle states are loop quantum gravity states of a single vertex. However, the graphs used to describe these states are usually not thought of as embedded in a manifold. Mathematically speaking, group field theory can be obtained from the quantization of certain field theories on groups, with the propagator and interaction terms describing valence of the vertices and details of the gluing.

在改变部分技术细节的前提下拓展形式体系的另一种可能是群场论 (GFT)(目前已有大量文献研究该理论的各个方面，关于从圈量子引力角度的介绍，参见 [107])。它的一个可能应用是作为几何中的多粒子理论类比；在这里，单粒子态的类比就是单个顶点的圈量子引力态。不过，描述这些态所用的图通常不被认为是嵌入在流形中的。从数学上说，群场论可以从群上某些场论的量子化得到，其传播子和相互作用项描述顶点的价和粘合的细节。

The Hilbert space $\mathcal{H}_{\text{GFT}}$ is by definition a Fock space. Geometric operators such as the area and volume operators (see sections "Quantum Geometry" and "Discrete Geometry") define one-particle operations that can be lifted to $\mathcal{H}_{\text{GFT}}$ via second quantization. Gauge invariance forces couplings between the states at different vertices that can elegantly create extended quantum geometries. Multi-particle operators can generate various dynamics on these geometries.

根据定义，希尔伯特空间 $\mathcal{H}_{\text{GFT}}$ 是福克空间。面积、体积这类几何算符 (参见章节“量子几何”与“离散几何”) 定义了单粒子操作，可通过二次量子化提升至 $\mathcal{H}_{\text{GFT}}$。规范不变性要求不同顶点处的态之间存在耦合，从而可以优美地构造出扩展量子几何。多粒子算符可以在这些几何上生成各类动力学。

There were also attempts to generalize the quantum representation of the holon-omy and flux variables to the differentiable or smooth category. The breakthrough was the extension of the integral (28) to the cylindrical functions defined by all the piecewise smooth curves [108]. For that purpose, a smooth generalization of an embedded graph was defined, namely, a web, whose edges can intersect and overlap infinitely many times, to form tassels. The resulting measure part and quantum representation of the cylindrical function observables give a complete and exact theory [109, 110]. The webs can be used to define a generalization of the spin-network states, with many properties of the spin-networks. However, the flux operators, the quantum area, and quantum volume are not well defined on the cylindrical functions that are given by webs of infinitely many self-intersections or intersections with the 2-surfaces used to define a given operator. In other words, the operators are not densely defined.

也有研究尝试将环绕通量变量的量子表示推广到可微或光滑范畴。研究的突破在于将积分 (28) 推广到了由所有分段光滑曲线定义的柱函数 [108]。为此，研究者定义了嵌入图的光滑推广——网络图 (web)，其边可以无数次相交、重叠，形成流苏结构。由此得到的测度部分与柱函数可观测量的量子表示构成了一个完备且精确的理论 [109, 110]。网络图可用于定义自旋网态的推广，保留自旋网的诸多性质。但对于由存在无限次自相交、或与定义算符所用二维曲面存在无限次相交的网络图给出的柱函数，通量算符、量子面积算符与量子体积算符都没有良好的定义，换句话说，这些算符不是稠定的。

Another interesting idea to generalize the class of differentiability of diffeomor-phisms are analytic diffeomorphisms beyond a finite number of points [111].

另一个推广微分同胚可微性类的有趣思路是，允许除有限个点外的解析微分同胚 [111]。

Diffeomorphisms are gauge symmetries of classical GR, which we quantize. That is why, according to the canonical approach, the manifold and its diffeomorphisms should pass naturally and in a scarified form into quantum theory. However, there is a radical, all-combinatorial approach to LQG, according to which the manifold does not exist as a background structure, and its structure is born in some other way, for example, through the polyhedral interpretation of vertices [112].

微分同胚是经典广义相对论的规范对称性，我们会对其进行量子化。因此根据正则方法，流形及其微分同胚应当以约化的形式自然进入量子理论。不过，存在一种彻底的全组合圈量子引力方法: 根据该观点，流形不作为背景结构存在，其结构是以其他方式生成的，例如通过顶点的多面体诠释生成 [112]。

Discrete Geometry

离散几何

There is a second, striking path to the quantum discreteness of geometrical operators. In this approach, one begins already with a discrete geometry, a geometry describable by a finite number of degrees of freedom, and making these degrees of freedom into a dynamical system works to understand the quantization of discrete geometry directly. There is a wealth of explicit results in this approach ranging from a general description of the phase space of shapes for convex polyhedra to explicit values for the quanta of volume of the simplest grain of space, a quantum tetrahedron.

几何算子的量子离散性还存在第二条引人注目的研究路径。在该方法中，我们从离散几何 (即可由有限个自由度描述的几何) 出发，将这些自由度构造成动力系统，从而直接研究离散几何的量子化。该方法已经得到了大量明确结果，范围涵盖凸多面体形状相空间的通用描述，到最简单空间单元——量子四面体——的体积量子具体取值。

In this section, we review this approach, frequently illustrating the methods and results using the quantum tetrahedron. The tetrahedral geometry and phase space, while the simplest, are already quite rich. However, we caution the reader that many aspects of the tetrahedral case are quite special. For example, the volume of a tetrahedron generates an integrable dynamical system, which is not the case for convex polyhedra with more facets. The full complexity of multifaceted polyhedra and higher-dimensional polytopes, such as the 4-simplex, and of complexes built up by gluing polyhedra together is of great interest. Indeed, a major open challenge in this setting is to build richly interacting networks of these discrete geometries that can be shown to be approximated by continuum regions of spacetime.

本节我们回顾该方法，并常以量子四面体为例说明相关方法与结果。四面体几何及其相空间虽最为简单，却已然相当丰富。但我们要提醒读者，四面体情形的诸多性质都非常特殊: 例如四面体的体积生成一个可积动力系统，而拥有更多面的凸多面体并不满足这一点。多面多面体与高维多胞体 (如 4-单形)，以及通过粘合多面体得到的复结构的完整复杂性都极具研究价值。事实上，该方向一个主要的开放挑战是构造这些离散几何的强相互作用网络，并证明它可以被连续时空区域近似。

Dynamical Discrete Geometries: Evolving Polyhedra

动力学离散几何: 演化多面体

In a static, weak field, the metric of spacetime is

在静态弱场条件下，时空度规为

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\left(dx^2+dy^2+dz^2\right). \tag{68}$$

In this limit, we can introduce a Newtonian gravitational field, given by $\mathbf{g} = -\nabla\Phi$, and in regions free of mass, this field satisfies $\boldsymbol{\nabla}\cdot\mathbf{g} = 0$. Using the divergence theorem and focusing on a region where the gravitational field can be taken to be constant, we have

在此极限下，我们可以引入由 $\mathbf{g} = -\nabla\Phi$ 给出的牛顿引力场，在无质量区域，该场满足 $\boldsymbol{\nabla}\cdot\mathbf{g} = 0$ 。利用散度定理并聚焦于一个引力场可视为恒定的区域，我们得到

$$\oint_{\mathcal{S}} \mathbf{g}\cdot\mathbf{da} = \mathbf{g}\cdot\oint_{\mathcal{S}}\mathbf{da} = 0 \tag{69}$$

This holds for any direction of the gravitational field $\mathbf{g}$, and so, for any small region enclosed by a surface $\mathcal{S}$, the oriented area of its boundary satisfies the closure condition

这对引力场 $\mathbf{g}$ 的任意方向都成立，因此对于任意被曲面 $\mathcal{S}$ 包围的小区域，其边界的定向面积满足闭合条件

$$\oint_{\mathcal{S}} \mathbf{da} = 0 \tag{70}$$

This is a fact about any spatial Euclidean geometry and will be quite central to what follows. It is interesting to see this fact emerging here as a consequence of the constant gravitational field or small region limit of the theory. The gravitational field is what determines the local inertial frames, and this identity holds in every sufficiently small spatial region of spacetime.

这是任意空间欧几里得几何都成立的结论，对后续推导至关重要。有趣的是，该结论在这里是作为理论中恒定引力场或小区域极限的推论出现的。引力场决定了局部惯性系，而这一等式在时空任意足够小的空间区域都成立。

In the special case of a region that is a spatial polyhedron with N facets, the integral (70) over the closed surface $\mathcal{S}$ breaks up into N pieces each of which has a fixed direction $\hat{n}_\ell$, and one obtains

对于区域是拥有 N 个面的空间多面体的特殊情况，闭合曲面 $\mathcal{S}$ 上的积分 (70) 可分解为 N 个部分，每个部分都有固定方向 $\hat{n}_\ell$ ，由此可得

$$\mathbf{A}_1 + \cdots + \mathbf{A}_N = 0 \tag{71}$$

where each area vector $\mathbf{A}_\ell := A_\ell \hat{n}_\ell$ (no sum) points normal to the ℓ th facet of the polyhedron and has magnitude A_ℓ equal to the area of this facet. (We use the index ℓ to label facets, and, of course, the area vectors $\mathbf{A}_\ell$ should not be confused with the Ashtekar connection A_a^i ; the relation with the Ashtekar variables will emerge below.) Surprisingly, exactly the identity (71) was leveraged by H. Minkowski, the same as of spacetime fame, to give a complete characterization of convex polyhedra at the close of the nineteenth century. Minkowski's theorem states that a set of N vectors $\{\mathbf{A}_\ell\}_{\ell=1}^N$ satisfying the closure (71) is in unique correspondence with a convex polyhedron of N facets up to overall rotation in space [45,113] (see Fig. 2 for an illustration of the case $N = 4$).

其中每个面积矢量 $\mathbf{A}_\ell := A_\ell \hat{n}_\ell$ (不求和) 指向多面体第 ℓ 个面的法向，其大小 A_ℓ 等于该面的面积。(我们用指标 ℓ 标记面，显然面积矢量 $\mathbf{A}_\ell$ 不可与阿西卡联络 A_a^i 混淆；它与阿西卡变量的关系我们会在后续说明。) 令人惊讶的是，正是等式 (71) 被因时空研究闻名的闵可夫斯基所用，在 19 世纪末给出了凸多面体的完整刻画。闵可夫斯基定理指出，一组满足闭合条件 (71) 的 N 个矢量 $\{\mathbf{A}_\ell\}_{\ell=1}^N$ ，与空间中拥有 N 个面的凸多面体一一对应，仅相差整体空间旋转 [45,113]($N = 4$ 情形的示例见图 2)。

As we will see, the discrete closure condition (71) is a powerful kinematical characterization of polyhedra. However, the true richness of the polyhedral approach emerges when we recognize the $\mathbf{A}_\ell$ as dynamical generators. In accordance with Kepler' s early insight, we will take each $\mathbf{A}_\ell$ to be an angular momentum, in the precise sense that they will be elements of the dual to the Lie algebra of $SU(2)$, $\mathbf{A}_\ell \in \mathfrak{su}^*(2) \cong \mathbb{R}^3$. There are myriad roots for this choice, ranging from Penrose's introduction of spin networks [43,44] to the fact that gravitational action integrals must always have the units of area. Here we will emphasize the fact that, with this choice, the closure (71) becomes a gravitational Gauss law: it imposes a constraint on the kinematical variables $\{\mathbf{A}_\ell\}_{\ell=1}^N$ that determines the shape of the N -faceted polyhedron, and in the same stroke, it generates the overall rotational gauge symmetry that signifies that this shape is insensitive to its rotational orientation in space. Let us turn now to establishing this.

我们将会看到，离散闭合条件 (71) 是多面体强有力的运动学刻画。然而，当我们认识到 $\mathbf{A}_\ell$ 是动力学生成元时，多面体方法真正的丰富性才得以展现。依照开普勒早年的洞见，我们将每个 $\mathbf{A}_\ell$ 视为角动量，准确来说它们是 $SU(2)$, $\mathbf{A}_\ell \in \mathfrak{su}^*(2) \cong \mathbb{R}^3$ 李代数对偶空间的元素。这一选择有诸多渊源，从彭罗斯引入自旋网络 [43,44]，到引力作用量积分必须具有面积量纲这一事实。在此我们需要强调的是，做出这一选择后，闭合条件 (71) 就成为引力高斯定律: 它对决定了 N 面多面体形状的运动学变量 $\{\mathbf{A}_\ell\}_{\ell=1}^N$ 施加约束，同时它生成整体旋转规范对称性，表明该形状不依赖其在空间中的旋转朝向。下面我们就来证明这一点。

We have emphasized that these angular momenta are in the dual to the Lie algebra because this is the deep reason that they come automatically equipped with a Poisson bracket, the Lie-Poisson bracket [115]. The details of this construction need not distract us as this bracket, for each $\mathbf{A}_\ell$, is just the familiar one of angular momenta, often expressed in terms of the Levi-Civita ε_{ijk} ; more generally, two functions of the full set of angular momenta, $f(\mathbf{A}_\ell)$ and $g(\mathbf{A}_\ell)$, have bracket given by the sum over facets of the familiar bracket:

我们已经强调，这些角动量处于李代数的对偶空间，因为这是它们自动配备泊松括号 (即李-泊松括号) 的深层原因 [115]。我们无需纠结该构造的细节，因为对每个 $\mathbf{A}_\ell$ ，这个括号就是我们熟悉的角动量括号，通常用列维-奇维塔 ε_{ijk} 表示；更一般地说，整套角动量的两个函数 $f(\mathbf{A}_\ell)$ 和 $g(\mathbf{A}_\ell)$ 的括号由各个小面的熟悉括号求和得到:

$$\{f,g\} = \sum_{\ell=1}^{N} \mathbf{A}_\ell \cdot \left(\frac{\partial f}{\partial \mathbf{A}_\ell} \times \frac{\partial g}{\partial \mathbf{A}_\ell} \right), \tag{72}$$

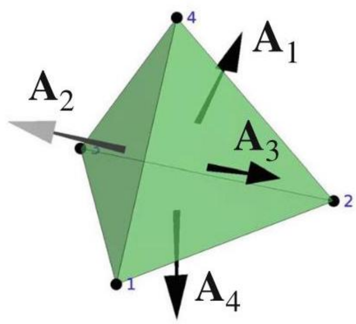


Fig. 2 A tetrahedron

图 2 一个四面体

together with its area vectors.

及其面积向量。

(Figure created with

(图片由

GIBBON [114])

GIBBON [114] 生成)

where $\partial f/\partial \mathbf{A}$ is a convenient shorthand for $\partial f/\partial A^i$ $(i=1,2,3)$. It is a quick check that for a fixed ℓ , this gives $\left\{A_\ell^i, A_\ell^j\right\} = \varepsilon^{ijk} A_\ell^k$.

其中 $\partial f/\partial \mathbf{A}$ 是 $\partial f/\partial A^i$ $(i=1,2,3)$ 的简便简写。不难验证，对固定的 ℓ ，该式给出 $\left\{A_\ell^i, A_\ell^j\right\} = \varepsilon^{ijk} A_\ell^k$ 。

This bracket has the property that any finite sum of the $\mathbf{A}_\ell$ generates a rotation of those vectors in the sum around the axis of the resultant. For example, $A_{12} := |\mathbf{A}_1 + \mathbf{A}_2|$ generates rotations of $\mathbf{A}_1$ and $\mathbf{A}_2$ around $\mathbf{A}_{12}$ and leaves $\{\mathbf{A}_3, \ldots, \mathbf{A}_N\}$ unchanged. More generally, $\hat{n} \cdot (\mathbf{A}_1 + \cdots + \mathbf{A}_N)$ is the diagonal generator of overall rotations of all of the vectors around the axis $\hat{n}$. This rotation does not, of course, change the closure of these vectors (Eq. (71)) or their relations to one another and, hence, leaves the shape of the polyhedron unchanged; this is the sense in which it generates a gauge transformation. The area vectors encode the intrinsic shape of the polyhedron and have a gauge symmetry with regard to its overall orientation in space.

该括号有这样的性质: $\mathbf{A}_\ell$ 的任意有限和都会生成这些向量绕合向量轴的旋转。例如，$A_{12} := |\mathbf{A}_1 + \mathbf{A}_2|$ 会生成 $\mathbf{A}_1$ 和 $\mathbf{A}_2$ 绕 $\mathbf{A}_{12}$ 的旋转，且保持 $\{\mathbf{A}_3, \ldots, \mathbf{A}_N\}$ 不变。更一般地，$\hat{n} \cdot (\mathbf{A}_1 + \cdots + \mathbf{A}_N)$ 是所有向量整体绕轴 $\hat{n}$ 旋转的对角生成元。当然，这种旋转不会改变这些向量的闭合性 (式 (71)), 也不会改变它们之间的相互关系，因此不会改变多面体的形状；从这个意义上说，它生成的是规范变换。面积向量编码了多面体的内蕴形状，并且关于其在空间中的整体取向具有规范对称性。

Summarizing, we have a remarkable triple of structures: (i) the geometry of any N -faceted polyhedron is described by the set of its area vectors $\{\mathbf{A}_\ell\}_{\ell=1}^N$ with each $\mathbf{A}_\ell \in \mathfrak{su}^*(2)$ and the full set satisfying the closure (71); (ii) these vectors have a Poisson bracket that can be used to generate dynamics; (iii) the magnitude of the closure relation (71) doubles as a gauge generator that focuses interest on the rotationally invariant properties of the shape of the polyhedron. These structures allow us to study the Hamiltonian dynamics of polyhedra: simply choose any rotationally invariant function $H(\mathbf{A}_\ell)$ of the $\{\mathbf{A}_\ell\}_{\ell=1}^N$ and treat it as a Hamiltonian generator of the flow

总结来说，我们得到了一组非常精妙的三重结构:(i) 任意 N 面多面体的几何由其面积向量集合 $\{\mathbf{A}_\ell\}_{\ell=1}^N$ 描述，每个 $\mathbf{A}_\ell \in \mathfrak{su}^*(2)$ 满足条件，整套向量满足闭合条件 (71)；(ii) 这些向量带有可用于生成动力学的泊松括号；(iii) 闭合关系 (71) 的模同时作为规范生成元，让我们将关注点聚焦在多面体形状的旋转不变性质上。这些结构允许我们研究多面体的哈密顿动力学: 只需任选一个 $\{\mathbf{A}_\ell\}_{\ell=1}^N$ 的旋转不变函数 $H(\mathbf{A}_\ell)$ ，将其作为流的哈密顿生成元即可

$$df/d\tau = \{f, H\}, \tag{73}$$

with τ a parameter along the flow. Rotationally invariant functions H of the $\mathbf{A}_\ell$ (other than $|\mathbf{A}_1 + \cdots + \mathbf{A}_N|$) will typically generate flows that change the shape of the polyhedron, which is encoded in the angles between the various $\mathbf{A}_\ell$, but not the number of facets or the closure of the polyhedron. Hence we obtain a dynamics for the discrete geometries described by N -faceted polyhedra.

其中 τ 是流沿的参数。除 $|\mathbf{A}_1+\cdots+\mathbf{A}_N|)$ 外，$\mathbf{A}_\ell$ 的旋转不变函数 H 通常会生成改变多面体形状的流，形状由不同 $\mathbf{A}_\ell$ 之间的夹角编码，但流不会改变面的数量，也不会改变多面体的封闭性。由此我们得到了由 N 面多面体描述的离散几何动力学。

We will call the Poisson space of N -faceted polyhedra that we have just described the space of polyhedra $\mathcal{P}_N$,

我们将刚刚描述的 N 面多面体的泊松空间称为多面体空间 $\mathcal{P}_N$ ，

$$\mathcal{P}_N := \left\{\mathbf{A}_\ell, \ell = 1, \ldots, N \mid \sum_\ell \mathbf{A}_\ell = 0\right\} / SO(3). \tag{74}$$

This space is given by the product of N copies of angular momentum space $\mathfrak{su}^*(2) \times \cdots \times \mathfrak{su}^*(2) = \left(\mathfrak{su}^*(2)\right)^N$, each copy isomorphic to $\mathbb{R}^3$, moded out by overall rotations of all of the vectors.

该空间由 N 份角动量空间 $\mathfrak{su}^*(2) \times \cdots \times \mathfrak{su}^*(2) = \left(\mathfrak{su}^*(2)\right)^N$ 的乘积构造得到，每一份都同构于 $\mathbb{R}^3$ ，再对所有向量的整体旋转做商处理。

The Phase Space of Polyhedra and Quantization

多面体的相空间与量子化

Above we have exhibited the space of N -faceted polyhedra as a Poisson space. However, it is an immediate consequence of the definition of the Poisson bracket in Eq. (72) that each of the magnitudes $A_\ell := |\mathbf{A}_\ell|$ is a Casimir function of this bracket, that is, $\{A_\ell, f\} = 0$ for all f . Thus, this bracket always preserves the magnitudes of the $\mathbf{A}_\ell$. Indeed, a foundational theorem on the structure of Poisson manifolds says that they are foliated by symplectic leaves with each leaf labelled by the value of a Casimir [116]. In our case, these leaves are the two-spheres picked out by the magnitudes A_ℓ . Each of these leaves is a symplectic manifold endowed with the Kirillov-Kostant-Soriau symplectic form, which in this case is given by $\omega = A_\ell \sin\theta d\theta \wedge d\phi = A_\ell d\Omega$ with (θ, ϕ) coordinates on the sphere and $d\Omega$ the solid angle [115, 117]. Thus, each of the angular momentum spheres is a standard phase space, and we can define a phase space of shapes for polyhedra.

上文我们已经将 N 面多面体的空间表示为一个泊松空间。然而，从式 (72) 中泊松括号的定义可以直接得到，每个量 $A_\ell := |\mathbf{A}_\ell|$ 都是该括号的卡西米尔函数，即对任意 f 都有 $\{A_\ell, f\} = 0$ 。因此，该泊松括号始终保持 $\mathbf{A}_\ell$ 的模长不变。实际上，关于泊松流形结构的一个基础定理指出，泊松流形可由辛叶层分，每个叶由卡西米尔函数的取值标记 [116]。在我们的情形中，这些叶就是由模长 A_ℓ 确定的二维球面。每个叶都是带基里洛夫-科斯坦特-索里奥辛形式的辛流形，此处该辛形式可写为 $\omega = A_\ell \sin\theta d\theta \wedge d\phi = A_\ell d\Omega$ ，其中 (θ, ϕ) 是球面上的坐标，$d\Omega$ 是立体角 [115, 117]。因此，每个角动量球面都是标准相空间，我们可以由此定义多面体形状的相空间。

Let Φ_N be the space of shapes of polyhedra with N facets of given areas A_l ,

设 Φ_N 为拥有给定面积 A_l 的 N 个面的多面体的形状空间，

$$\Phi_N = \left\{ \mathbf{A}_\ell, \ell = 1, \dots, N \mid \sum_\ell \mathbf{A}_\ell = 0, |\mathbf{A}_\ell| = A_\ell \right\} / SO(3). \tag{75}$$

This space is given by the product of N spheres $S^2 \times \cdots \times S^2 = (S^2)^N$, obtained by fixing the magnitudes $A_1, \dots, A_N$, and moded out by overall rotations of all of the vectors. Its dimension is $\dim \Phi_N = 2N - 6$, which is determined by the dimension of this collection of spheres, $2N$, minus three, for the conditions $\sum_\ell \mathbf{A}_\ell = 0$, and minus three more for the division by overall rotations. This phase space was introduced and studied by M. Kapovich and J. J. Millson in the somewhat different context of linkages [118]. The advantage of this phase space is that the areas of the facets can be regarded as a fixed, parametric dependence during calculations. Nonetheless, we will freely transition back and forth between the Poisson and symplectic pictures, adopting whichever is more convenient for the discussion at hand.

该空间由固定模长 $A_1, \dots, A_N$ 得到的 N 个球面 $S^2 \times \cdots \times S^2 = (S^2)^N$ 作直积后，再对所有向量整体旋转作商得到。其维数为 $\dim \Phi_N = 2N - 6$ ，由球面集合的维数 $2N$ 减去满足条件 $\sum_\ell \mathbf{A}_\ell = 0$ 的 3 个维度，再减去整体旋转作商的 3 个维度得到。该相空间最早由 M. 卡波维奇与 J.J. 米尔森在连杆机构的不同研究背景中引入并研究 [118]。这个相空间的优势在于，计算过程中可以将多面体面的面积视为固定参数。尽管如此，我们仍可在泊松图景与辛图景之间自由转换，根据当前讨论选用更方便的一种。

A First Interlude on Quantization: The Quantization of Area and Spin Network Nodes and the Meaning of Quantizing Grains of Space

量子化第一插曲: 面积量子化、自旋网络节点与空间量子化颗粒的意义

Our first result on the quantization of discrete geometries follows immediately from our choice of dynamical variables, the $\mathbf{A} \in \mathfrak{su}^*(2)$. We associate a Hilbert space $\mathcal{H}_j$ with each facet of the polyhedron, or what is equivalent, with the edge of a spin network graph that crosses this face transversally. This is a carrier space for a unitary irrep of $SU(2)$, so that $\dim \mathcal{H}_j = 2j+1$. The $\mathbf{A}$ is a standard angular momentum variable, and the quantization of its magnitude squared is given by

我们关于离散几何量子化的首个结果直接来自我们对动力学变量 $\mathbf{A} \in \mathfrak{su}^*(2)$ 的选择。我们将希尔伯特空间 $\mathcal{H}_j$ 与多面体的每个小面对应起来——换句话说，也就是与横截穿过该面的自旋网络图的边对应起来。这是 $SU(2)$ 的一个幺正不可约表示的承载空间，因此有 $\dim \mathcal{H}_j = 2j + 1$ 。$\mathbf{A}$ 是标准角动量变量，其模平方的量子化可表示为

$$\widehat{A}^2 \big| jm\rangle = (\hbar\kappa\gamma)^2 j(j+1) \big| jm\rangle, \tag{76}$$

where $|jm\rangle$ is a basis of $\mathcal{H}_j$; the physical scale of the quantization is the Planck area $a_P = \hbar\kappa$; and the gap between zero and the lowest lying area eigenstate has been parameterized by the Barbero-Immirzi parameter $\gamma > 0$. This remarkable result ties this choice of variables to a physical prediction: a Planck-scale discrete spectrum for the areas that connect neighboring regions of space.

其中 $|jm\rangle$ 是 $\mathcal{H}_j$ 的一组基；量子化的物理标度是普朗克面积 $a_P = \hbar\kappa$；零与最低能量本征态之间的间隙由巴贝罗-伊米爾齐参数 $\gamma > 0$ 参数化。这一显著结果将该变量选择与一个物理预言联系起来: 连接空间相邻区域的面积具有普朗克尺度的离散谱。

The m quantum number in the states introduced here, the eigenvalue of $\widehat{A}_z$, exhibits an orientation dependence of these states. This parallels the orientation dependence of any one of the area vectors $\mathbf{A}_\ell$. Only upon consideration of the full set of area vectors, satisfying the closure condition, is it that orientation independence is possible. Similarly, in the quantum case, the key to achieving orientation independence is to form a new state by combining facet states together using a rotationally invariant tensor that lives inside the product of the facet Hilbert spaces. Thus, we introduce the subspace of the tensor product Hilbert space $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ that is invariant under the global action of $SU(2)$ rotations

本文引入的态中，m 量子数即 $\widehat{A}_z$ 的本征值，展现出这些态的取向相关性。这与任意一个面积矢量 $\mathbf{A}_\ell$ 的取向相关性一致。只有考虑满足闭合条件的全套面积矢量，才可能实现取向无关性。类似地，在量子情形下，实现取向无关性的关键是，利用存在于小面希尔伯特空间乘积中的旋转不变张量，将各个小面态组合成新态。因此，我们在张量积希尔伯特空间 $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ 中引入在 $SU(2)$ 旋转整体作用下不变的子空间

$$\mathcal{H}_N = \mathrm{Inv}\left(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}\right). \tag{77}$$

We call an invariant state, $|\iota\rangle$, of this space an intertwiner and $\mathcal{H}_N$ the space of intertwiners. Each state $|\iota\rangle \in \mathcal{H}_N$ can be expanded in a $|jm\rangle$ basis for each facet,

我们将该空间中的不变态 $|\iota\rangle$ 称为缠结算子，并称 $\mathcal{H}_N$ 为缠结算子空间。每个态对应每个小面的基 $|\iota\rangle \in \mathcal{H}_N$ can be expanded in a $|jm\rangle$,

$$\Big|\iota\rangle = \sum_{m's} \iota^{m_1 \cdots m_N} \Big| j_1 m_1\rangle \cdots |j_N m_N\rangle, \tag{78}$$

and their components $\iota^{m_1 \cdots m_N}$ transform as a tensor under $SU(2)$ transformations (cf. section "Ashtekar-Lewandowski Representation"). The defining condition that the states $|\iota\rangle$ are rotationally invariant can be expressed as the invariance of these components under the diagonal action of $SU(2)$.

其分量 $\iota^{m_1 \cdots m_N}$ 在 $SU(2)$ 变换下按张量变换 (参见“Ashtekar-Lewandowski 表示”一节)。态 $|\iota\rangle$ 具有旋转不变性的定义条件，可以表述为这些分量在 $SU(2)$ 的对角作用下不变。

Different choices of basis in the Hilbert space $\mathcal{H}_N$ emphasize different aspects of the resulting states; here we will focus on choices that highlight different aspects of the quantization of polyhedra [45]. That is, we will emphasize the geometry of quantum polyhedra. The literature explores a rich set of alternative choices, each with its own character and advantages [54, 119-121]. The focus on quantum polyhedra here is for specificity and concreteness and is made to offer a route into this literature. That said, the reader will do well to remember that there are more ways of viewing the intertwiner space $\mathcal{H}_N$ than can be covered here.

希尔伯特空间 $\mathcal{H}_N$ 中不同的基选择会突出最终得到的态的不同方面；本文我们将聚焦于凸显多面体量子化不同侧面的选择 [45]，也就是说，我们将重点讨论量子多面体的几何。文献中已经研究了丰富的可选基选择，每种都有其自身特点与优势 [54, 119-121]。本文聚焦量子多面体是为了明确具体，为进入该领域文献提供一条入门路径。尽管如此，读者仍需注意，看待缠结算子空间 $\mathcal{H}_N$ 的方式远多于本文所能涵盖的内容。

The Hilbert space $\mathcal{H}_N$ is associated with an N -valent node of a spin network. Indeed, these are the gauge-invariant building blocks out of which the Hilbert space associated with a spin network graph Γ is built. With this connection made, it is now possible to tie together the results of the first half of this chapter to the discussion of this section. For simplicity, consider a single 4-valent node of a spin network; as discussed above, this node corresponds classically to a tetrahedron. The area vectors $\{\mathbf{A}_\ell\}_{\ell=1}^4$ discussed here are the fluxes $P_{S,f}$, defined at (14), where the integration surfaces S are each of the facets of tetrahedron. The function f , valued in the Lie algebra, gives the direction of the normal to the facet $\hat{n}_\ell$, and the rich non-commutativity of the components of each $\mathbf{A}_\ell$ can be seen as a consequence of the need to parallel transport the choice of internal frame at the center of the tetrahedron out to the facet along the path p using the holonomies h_p , defined at (12) or, alternatively, as a consequence of the gauge-invariant smearing along the facet using f (see [122] and [123], respectively). Thus, the $\mathbf{A}_\ell$ together with the closure (71) and the Poisson structure (72) are a discrete summary of the holonomy-flux algebra; parallel statements to all of these can be made about the quantum operators $\widehat{\mathbf{A}}_\ell$ and $\widehat{P}_{S,f}, \left(\widehat{h}_p\right)^a{}_b$, defined at (22).

希尔伯特空间 $\mathcal{H}_N$ 与自旋网络的一个 N 价节点相关联。实际上，这些是规范不变的构造块，与自旋网络图 Γ 关联的希尔伯特空间正是由它们构造而来。建立这一关联后，我们现在就可以将本章前半部分的结论与本节的讨论联系起来。为简化起见，考虑自旋网络的单个四价节点；正如上文所讨论的，该节点经典地对应一个四面体。此处讨论的面积向量 $\{\mathbf{A}_\ell\}_{\ell=1}^4$ 就是式 (14) 中定义的通量 $P_{S,f}$ ，其中积分曲面 S 分别是四面体的每个面。取值于李代数的函数 f 给出面 $\hat{n}_\ell$ 的法向方向，每个 $\mathbf{A}_\ell$ 各分量之间丰富的非对易性可以看作是以下操作的结果: 利用式 (12) 定义的整体规范变换 h_p ，沿路径 p 将四面体中心内部参考架的选择平行移动到该面；或者也可以看作是利用 f 沿面进行规范不变涂抹的结果 (分别参见文献 [122] 和 [123])。因此，$\mathbf{A}_\ell$ 与闭合条件 (71)、泊松结构 (72) 一起构成了整体流-通量代数的离散概括；对于式 (22) 定义的量子算符 $\widehat{\mathbf{A}}_\ell$ 和 $\widehat{P}_{S,f}, \left(\widehat{h}_p\right)^a{}_b$ ，所有这些结论都有平行的量子对应。

More generally, a picture of the quantum geometry of space is emerging: it can be seen as a collection of quantum polyhedra, invariant under local choices of frame, and glued along their equal area facets. Holonomies capture the curvature as collections of these polyhedra are traversed in closed paths. This rich, intuitive, and mathematical picture is a consequence of the surprisingly simple quantization discussed above and is the foundation of the quantum geometry of loop quantum gravity. However, its interpretation is subtle, and it is worth spending some time clarifying the conceptual content of this result.

更一般地，空间量子几何的图景正在显现: 空间可以看作是量子多面体的集合，这些多面体在局域参考架选取下不变，并沿着面积相等的面粘合在一起。当沿闭合路径穿越这些多面体的集合时，整体流就捕捉到了曲率。这一内容丰富、直观且严谨的数学图景，是上文所讨论的出人意料的简单量子化过程的结果，也是圈量子引力量子几何的基础。不过，它的诠释十分微妙，值得我们花时间澄清这一结果的概念内涵。

The concreteness of the quantum polyhedral picture can be deceptive unless emphasis is put on two aspects of the quantization procedure being considered: the truncation to a finite number of degrees of freedom of the gravitational field and the meaning of the modifier quantum. We take up each of these points in sequence.

如果不强调我们所考虑的量子化过程的两个方面，量子多面体图景的具体性反而会造成误导: 这两个方面就是引力场自由度到有限数量的截断，以及修饰词「量子」的含义。下面我们依次讨论每一点。

As emphasized in the preamble to this section, loop quantum gravity on a fixed graph Γ is a truncation of the infinite number of degrees of freedom of the gravitational field down to a finite subset. Rovelli and Speziale [124] have pointed out that this is analogous to sampling a function $f(x)$ at a finite number of points $f_n = f(x_n)$ for $n \in \{1, \dots, 7\}$, say. Given this finite sample, it is not possible to reconstruct all of the information contained in the function $f(x)$ (see Fig. 3, after [124]). Nonetheless, much like in signal processing, there are definite insights that can be gained from various interpolations of these data. The polyhedra considered here are one such interpolation choice in quantum gravity; depending on how they are glued together, they are like the piecewise linear interpolation of the third panel, as in Regge calculus [51], or the piecewise flat interpolation of the fourth panel, as in twisted geometries [54] (It is the nature of these polyhedra as an interpolating scheme that leaves researchers unconcerned about the rigidity of the convexity assumption in Minkowski's theorem introduced above.). It is best not to confuse this interpolation choice with a fundamental statement about nature. The claim in loop quantum gravity is not that there are little polyhedral pieces that make up space but that the discrete geometry of polyhedra can model a finite number of the degrees of freedom of the gravitational field. And it is remarkable that, just as in the continuum, these degrees of freedom, the ones of discrete polyhedra, can be used to describe the dynamical system that is the evolving geometry of this approximation of a spatial region.

正如本节序言所强调的，固定图 Γ 上的圈量子引力是将引力场无穷多自由度截断为有限子集。罗韦利和斯佩齐亚 [124] 已指出，这就好比对函数 $f(x)$ 在有限个点 $f_n = f(x_n)$ 上进行采样用于 $n \in \{1, \dots, 7\}$。仅依靠这个有限样本，无法还原函数 $f(x)$ 包含的全部信息 (参见图 3，引自文献 [124])。尽管如此，和信号处理的情况非常相似，对这些数据做不同插值仍能得到明确的结论。本文讨论的多面体就是量子引力中这类插值选择之一；根据粘合方式的不同，它们就像第三幅图中的分段线性插值，对应里奇微积分 [51]，或是第四幅图中的分段平坦插值，对应扭曲几何 [54](正是这些多面体作为插值方案的性质，让研究者无需担心前文引入的闵可夫斯基定理中凸性假设的刚性问题)。最好不要将这种插值选择误认为关于自然本质的基础论断。圈量子引力并没有主张空间由微小的多面体块拼接而成，而是说多面体的离散几何可以对引力场的有限自由度建模。值得注意的是，和连续统情形一样，离散多面体的这些自由度可以用来描述动力学系统，即空间区域近似下演化的几何。

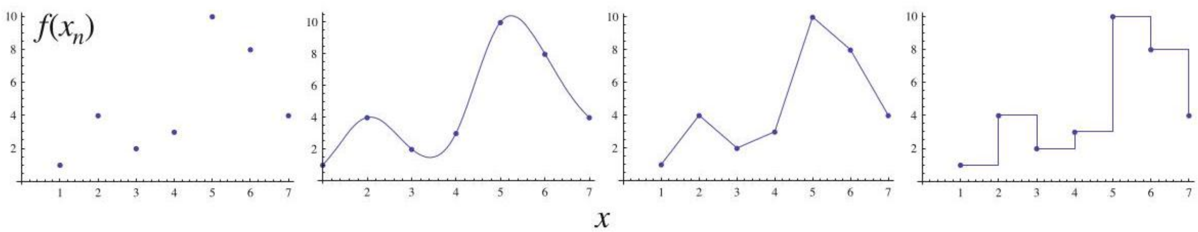


Fig. 3 A set of finite samples of a function $f(x)$ together with three different interpolations of the samples: polynomial, piecewise linear, and piecewise flat. These interpolation methods are analogous to mode sampling in cosmology, Regge geometries [51], and twisted geometries [54]

> 图 3 函数 $f(x)$ 的一组有限样本，以及对样本的三种不同插值: 多项式插值、分段线性插值和分段平坦插值。这些插值方法分别对应宇宙学中的模式采样、里奇几何 [51] 和扭曲几何 [54]

A result that must be emphasized in immediate counterpoint is that there is a claim of fundamental, physical discreteness here nonetheless. It is the spectral discreteness of the geometrical operators in loop quantum gravity. The operator that probes the area of a polyhedral facet, or, equivalently, of a spin network edge, has a discrete spectrum (Eq. (76)). Below we will see that the volume spectrum of a tetrahedron is also discrete. These spectra are a physical prediction of loop quantum gravity: if we were able to experimentally probe Planck-scale regions of space, the theory predicts that we would measure this discreteness directly. Thus, the meaning one should attach to the notion of a quantum grain of space is not that of a particular polyhedron (or any other model of part of the grain's degrees of freedom) but rather the insight that if one were to measure aspects of the grain's geometry, one would obtain spectral, quantum results.

> 与之形成直接对比、必须强调的一个结论是: 圈量子引力确实主张存在基础的物理离散性，即几何算符的谱离散性。探测多面体面 (等价于自旋网络边) 面积的算符具有离散谱 (式 (76))。下文我们会看到，四面体的体积谱也是离散的。这些谱是圈量子引力的物理预言: 如果我们能通过实验探测普朗克尺度的空间区域，该理论预言我们将直接观测到这种离散性。因此，人们对空间量子颗粒这一概念应当赋予的含义，不是它对应某个特定多面体 (或任何其他颗粒部分自由度的模型)，而是要认识到: 如果测量颗粒几何的相关性质，就会得到量子谱离散的结果。

Another way of approaching this point, which also connects with the semiclassical methods discussed below, is to say that a quantum polyhedron is fuzzy. That is, we understand that when we refer to an energy eigenstate of an harmonic oscillator, this state is spread out over all the classical configurations that have this energy; the state does not correspond to a single, classical state but is probabilistically spread out over all of them. In a semiclassical picture, the classical phase space of the harmonic oscillator is $\mathbb{R}^2$ with coordinates (x, p) and symplectic 2-form $\omega = dp \wedge dx$. The level set $H(x, p) = 1/2\left(p^2 + x^2\right) = E$ is a circle and has the property that the pullback of the symplectic form to this set vanishes; we say that any submanifold of the phase space with this property is Lagrangian. The oscillator's energy eigenstates can be seen as quantizing these extended Lagrangian submanifolds of the phase space. Similarly, we will show below that a quantum polyhedron, e.g., a volume eigenstate of the tetrahedron, can be seen as spread out over the classical shapes that have that volume. In this sense, their shapes are quantum mechanically fuzzy. Before giving a detailed treatment of this perspective, let us complete our discussion of the phase space of shapes of polyhedra.

理解这一点的另一种方式，同时也和下文讨论的半经典方法相关，就是说量子多面体是模糊的。也就是说，我们都知道当我们提到谐振子的能量本征态时，这个态弥散在所有具有该能量的经典构型上；该态不对应单个经典态，而是以概率弥散在所有经典构型上。在半经典图像中，谐振子的经典相空间是 $\mathbb{R}^2$ ，坐标为 (x, p) ，辛 2 形式为 $\omega = dp \wedge dx$ 。等能级集合 $H(x, p) = 1/2\left(p^2 + x^2\right) = E$ 是一个圆，且辛形式拉回到该集合上为零；我们将相空间中满足这一性质的任意子流形称为拉格朗日子流形。谐振子的能量本征态可以看作是对相空间中这些扩展拉格朗日子流形的量子化。类似地，我们下文将证明，量子多面体 (例如四面体的体积本征态) 可以看作弥散在所有具有该体积的经典形状上。在这个意义上，它们的形状在量子力学上是模糊的。在详细讨论这一视角之前，我们先完成对多面体形状相空间的讨论。

Action-Angle Coordinates

作用量-角度坐标

One of the remarkable aspects of the space of shapes of polyhedra Φ_N that was recognized by Kapovich and Millson [118] is that it comes naturally endowed with action-angle coordinates. (See, e.g., [125], for an introduction to this special class of coordinates on phase space.) On Φ_N , these coordinates are most easily constructed geometrically. Fix the magnitudes of a set of area vectors $\{\mathbf{A}_\ell\}_{\ell=1}^N$, and consider a special configuration of these N vectors in which they are all coplanar. This special configuration can be viewed as an N -sided polygon living in a plane of $\mathbb{R}^3$ (see Fig. 4). Now, consider the set of vectors $\mathbf{I}_k = \sum_{l=1}^{k+1} \mathbf{A}_l$, where $k = 1, \ldots, N-3$, which define a set of diagonals of this polygon. We define the coordinate ϕ_k as the angle between the vectors $\mathbf{I}_k \times \mathbf{A}_{k+1}$ and $\mathbf{I}_k \times \mathbf{A}_{k+2}$ and the action variable $I_k = |\mathbf{I}_k|$ as the norm of the vector $\mathbf{I}_k$.

卡波维奇 (Kapovich) 和米尔森 (Millson)[118] 发现，多面体形状空间 Φ_N 一个值得注意的性质是，它天然就带有作用量-角度坐标。(关于相空间中这类特殊坐标的介绍可参见例如文献 [125]。) 在 Φ_N 上，这些坐标可以非常简便地通过几何方法构造。固定一组面积矢量 $\{\mathbf{A}_\ell\}_{\ell=1}^N$ 的模长，然后考虑这 N 个矢量的一个特殊构型: 所有矢量共面。这个特殊构型可以看作是位于 $\mathbb{R}^3$ 平面内的 N 边形 (参见图 4)。现在，考虑由对角线定义的矢量集合 $\mathbf{I}_k = \sum_{l=1}^{k+1} \mathbf{A}_l$ ，其中 $k = 1, \ldots, N-3$ 。我们将坐标 ϕ_k 定义为矢量 $\mathbf{I}_k \times \mathbf{A}_{k+1}$ 与 $\mathbf{I}_k \times \mathbf{A}_{k+2}$ 之间的夹角，将作用量变量 $I_k = |\mathbf{I}_k|$ 定义为矢量 $\mathbf{I}_k$ 的模长。

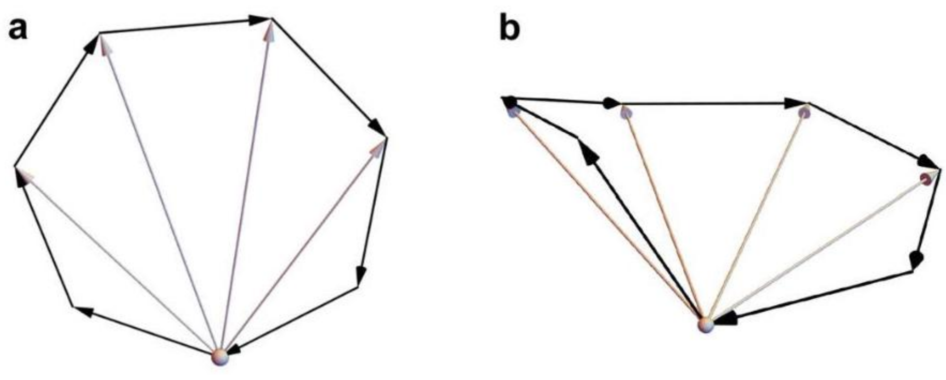


Fig. 4 (a) An example of a planar configuration of seven area vectors, black arrows, satisfying closure. The magnitudes of the four gray vectors define the actions $\{I_k\}_{k=1}^4$ that generate the bending flows. (b) The result of following the I_1 flow $\pi/4$ radians and then the I_2 flow $\pi/6$ radians

图 4(a) 满足闭合条件的七个面积矢量的平面构型示例，黑色箭头为面积矢量。四个灰色矢量的模长确定了生成弯曲流的作用量 $\{I_k\}_{k=1}^4$ 。(b) 先沿 I_1 流演化 $\pi/4$ 弧度、再沿 I_2 流演化 $\pi/6$ 弧度后的结果

The action I_k generates rotations of each area vector appearing in its definition around the corresponding diagonal $\mathbf{I}_k$. These transformations are referred to as bending flows (Fig. 4, panel (b)). In a slight generalization of the notion, we continue to refer to the resulting configuration of the area vectors as a polygon in $\mathbb{R}^3$, even though it is no longer planar. Furthermore, up to overall rotations of all of the vectors, every polygon of $\mathbb{R}^3$, with the given values of the $\{A_\ell\}_{\ell=1}^N$, can be reached by varying the I_k over their finite range of values and considering the bending flows they generate [118]. Thus, the set of $\{(\phi_k, I_k)\}_{k=1}^{N-3}$ pairs provide global coordinates on the $(2N-6)$ -dimensional phase space Φ_N . From (72), it follows that these are canonically conjugate variables, $\{\phi_k, I_{k'}\} = \delta_{kk'}$, and so we have a set of action-angle variables.

作用量 I_k 会使其定义中涉及的每个面积矢量绕对应对角线 $\mathbf{I}_k$ 转动。这类变换被称为弯曲流 (图 4(b))。对该概念稍加推广后，我们仍将面积向量得到的构型称为 $\mathbb{R}^3$ 中的多边形，即便它已经不再是平面构型。此外，固定 $\{A_\ell\}_{\ell=1}^N$ 的取值后，在所有向量整体转动的范围内，通过改变 I_k 在有限取值范围内的取值并考虑它们生成的弯曲流，可以得到任意一个 $\mathbb{R}^3$ 中的多边形 [118]。因此，$\{(\phi_k, I_k)\}_{k=1}^{N-3}$ 对构成了 $(2N-6)$ 维相空间 Φ_N 上的全局坐标。由式 (72) 可得，它们是正则共轭变量，即满足 $\{\phi_k, I_{k'}\} = \delta_{kk'}$ ，因此我们得到了一组作用量-角度变量。

The simplest possible polyhedron, the $N = 4$ tetrahedron (Fig. 2), to which we turn now, provides the perfect example for unpacking the details of all of the structures and correspondences that we have just reviewed.

最简单的多面体是 $N = 4$ 四面体，我们现在就来讨论它——它是一个绝佳的例子，可以帮我们拆解清楚刚才介绍的所有结构与对应关系的细节。

Quantum Tetrahedra

量子四面体

The Classical Geometry of the Space of Shapes of Tetrahedra

四面体形状空间的经典几何

The simplest discrete geometry that you can build in three-dimensional space is a tetrahedron. The action-angle variables and bending flow are particularly simple in this case: for $N = 4$, the space of shapes, Φ_4 , is two-dimensional, and its single action variable is $I_1 := A_{12} = |\mathbf{A}_1 + \mathbf{A}_2|$. This action generates the bending flow that increments the angle ϕ between the two wings of the quadrilateral formed by the closure of the four area vectors $\mathbf{A}_1, \ldots, \mathbf{A}_4$ (see the first panel of Fig. 5).

三维空间中可构造的最简单离散几何就是四面体。作用角变量与弯曲流在此情形下格外简单: 对形状空间 $N=4$ 而言，Φ_4 是二维的，其唯一作用变量为 $I_1 := A_{12} = |\mathbf{A}_1 + \mathbf{A}_2|$ 。该作用生成弯曲流，使四个面积向量 $\mathbf{A}_1, \ldots, \mathbf{A}_4$ 闭合形成的四边形两翼之间的夹角 ϕ 增量变化 (参见图 5 第一幅分图)。

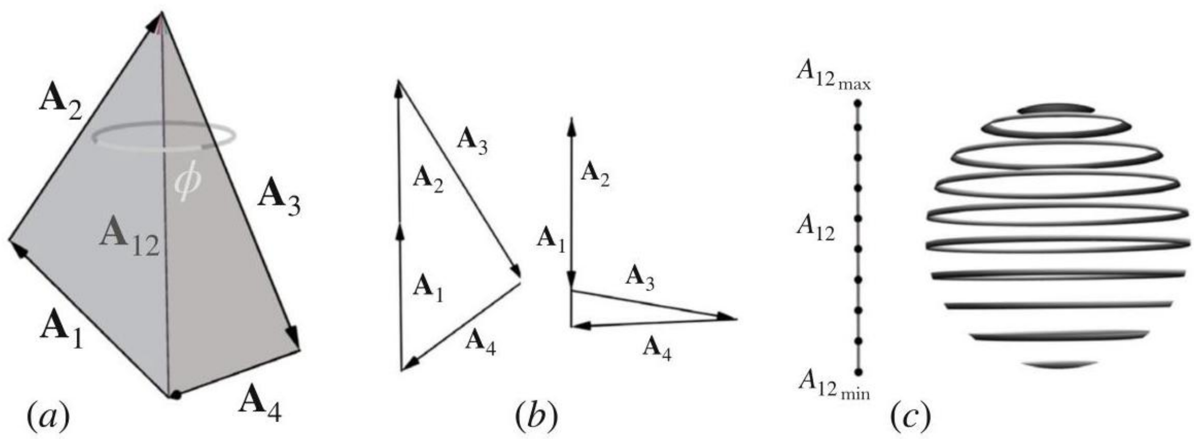


Fig. 5 (a) A non-planar configuration of the four area vectors of a tetrahedron. The action coordinate of the space of shapes Φ_4 is $A_{12} = |\mathbf{A}_{12}|$. The conjugate angle, ϕ , is given by the angle between the two planar wings in gray, spanned by $\{\mathbf{A}_1, \mathbf{A}_2\}$ and by $\{\mathbf{A}_3, \mathbf{A}_4\}$, respectively. (b) The bending flow on the vector configurations that achieve the maximum and minimum values of $\mathbf{A}_{12} = \mathbf{A}_1 + \mathbf{A}_2$ generates a rotationally equivalent configuration of vectors and hence corresponds to a single point of the phase space; the north and south pole of the phase space sphere depicted in the last panel. (c) Each point of the interval of allowed values for the action $A_{12} \in [A_{12\,\min}, A_{12\,\max}]$ corresponds to a circle of rotationally distinct vector configurations, except at the endpoints of the interval. The collection of all of these circles and the single points at the extremal values gives the phase space of shapes of tetrahedra, which is topologically a sphere

图 5 (a) 四面体四个面积向量的非平面构型。形状空间 Φ_4 的作用坐标为 $A_{12} = |\mathbf{A}_{12}|$ ，其共轭角 ϕ 即分别由 $\{\mathbf{A}_1, \mathbf{A}_2\}$ 和 $\{\mathbf{A}_3, \mathbf{A}_4\}$ 张成的两个灰色平面翼之间的夹角。(b) 作用量 $\mathbf{A}_{12} = \mathbf{A}_1 + \mathbf{A}_2$ 取最大和最小值时向量构型上的弯曲流，生成旋转等价的向量构型，对应相空间中的单个点，即最后一幅分图中相空间球面的北极与南极。(c) 作用量 $A_{12} \in [A_{12\,\min}, A_{12\,\max}]$ 允许取值区间内的每个点，除区间端点外，都对应一个旋转不等价向量构型构成的圆。所有这些圆加上极值处的单个点共同构成四面体形状的相空间，拓扑上是一个球面

In fact, the action-angle coordinates help to uncover the topology of the space of shapes of tetrahedra Φ_4 . For a generic value of A_{12} , the bending flow sweeps out a full circle's worth of area vector configurations. Each of these configurations is distinct in the sense that no overall rotation of the four area vectors will transform one of these configurations into another. However, the action variable A_{12} has a finite range, spanning the interval $\max(|A_1 - A_2|, |A_3 - A_4|) \leq A_{12} \leq \min(A_1 + A_2, A_3 + A_4)$, and the end points of this range are special. In panel (b) of Fig. 5, the generic configurations of the area vectors at the ends of the range are illustrated. The bending flows of these special "flag" configurations no longer generate rotationally distinct configurations up to overall rotations of all four vectors. This means that at the extreme values of A_{12} , there is no longer a full circle of configurations but only a single point. Thus, topologically, the space Φ_4 is given by the Cartesian product of an interval with a circle that degenerates into a point at the two end points of the interval; this is precisely the topology of a sphere (Fig. 5, panel (c)).

事实上，作用角坐标有助于揭示四面体形状空间 Φ_4 的拓扑结构。对于 A_{12} 的一般取值，弯曲流会扫过整整一圈面积向量构型。每个构型都是不同的，不存在四个面积向量的整体旋转能将其中一个构型转变为另一个。然而，作用变量 A_{12} 的取值范围有限，落在区间 $\max(|A_1 - A_2|, |A_3 - A_4|) \leq A_{12} \leq \min(A_1 + A_2, A_3 + A_4)$ 内，该区间的端点是特殊的。图 5 的 (b) 分图展示了取值区间端点处面积向量的一般构型。这些特殊“旗状”构型的弯曲流，在考虑四个向量的整体旋转后，不再生成旋转不同的构型。这意味着在 A_{12} 的极值处，不存在整圈的构型，仅存在单个点。因此从拓扑上说，空间 Φ_4 是区间与圆的笛卡尔积，且圆在区间的两个端点退化为点；这恰好是球面的拓扑结构 (图 5, (c) 分图)。

Having discovered the topology and symplectic structure on the space of shapes of tetrahedra, our main goal in this section will be to introduce a geometrically motivated choice of intertwiner on this space and to quantize it. This is a key step toward the development of the discrete-geometry path integrals for gravity called spin foam models, such as the Barret-Crane model [126,127] and the EPRL model discussed in detail in PChap. 86, ”Spin Foams: Foundations.” As stressed above, such an inter-twiner should be a rotationally invariant operator that lives on the space of shapes. Geometrically a clearly motivated choice would be the volume of the tetrahedron [128,129]. Remarkably, for a Euclidean tetrahedron, the classical volume, V , also has a simple rotationally-invariant expression in terms of the area vectors

在已经了解四面体形状空间的拓扑结构与辛结构之后, 本节我们的主要目标是在该空间上引入一个具有几何动机的缠结算子选择，并对其进行量子化。这是构建引力离散几何路径积分 (即自旋泡沫模型，例如巴雷特-克莱恩模型 [126,127] 和第 86 章“自旋泡沫: 基础”中详细讨论的 EPRL 模型) 的关键步骤。正如上文强调，这样的缠结算子应当是定义在形状空间上的旋转不变算子。从几何角度看，一个动机明确的选择是四面体的体积 [128,129]。值得注意的是，对于欧几里得四面体，经典体积 V 也可以用面积向量表示为一个简单的旋转不变表达式

$$V^2 = \frac{2}{9}\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)\,. \tag{79}$$

This can be quickly checked by expressing the area vectors in terms of cross products of the vectors that run along the edges of the tetrahedron. At first glance, this expression seems to favor three of the four area vectors, but the closure relation (Eq. (71)) allows you to rewrite this formula in terms of any three. The left panel of Fig. 6 illustrates the space of shapes Φ_4 , a typical point corresponding to a tetrahedron in inset (i) , and some example contours of constant volume V .

我们可以将面积向量用四面体棱边向量的叉积表示，很快就能验证这一点。乍看之下，该表达式似乎更偏向四个面积向量中的三个，但利用闭关系 (式 (71)) 可以将该公式改写为任意三个面积向量的形式。图 6 左面板展示了形状空间 Φ_4 ，插图 (i) 中是对应于一个典型点的四面体，还给出了几个等体积 V 轮廓线的示例。

Before proceeding to the quantization of this volume, there are three entwined subtleties about the classical phase space of shapes to clarify. First, consider area vectors that are coplanar, precisely the configurations used above to introduce the action-angle coordinates. These planar configurations of area vectors certainly lead to $V = 0$; however, their interpretation is tricky. One might have expected that they all corresponded to planar configurations of the tetrahedron itself, like a tent squashed down onto its floor, but most of them do not. Observe that vectors pointing along the edges of a tetrahedron can be obtained by taking the cross-product of any two area vectors. For a planar configuration of the area vectors, all of these cross-products are collinear, and the tetrahedron has degenerated beyond planarity into a one-dimensional ”pencil-like” tetrahedron. Note that while these tetrahedra are degenerate, even a small deformation away from one of them is not, and it is useful to keep these limit points in the phase space. Generically, the great circle on the spherical space of shapes obtained by merging the longitudes $\phi = 0$ and $\phi = \pi$, which is all the configurations with $V = 0$, consists completely of pencil-like tetrahedra. The exception is when the space of shapes includes planar tetrahedra, discussed in the third point below.

在对该体积进行量子化之前，我们需要澄清形状经典相空间三个相互关联的微妙问题。首先，考虑共面的面积向量，这正是我们上文引入作用量-角坐标时用到的构型。这些共面面积向量构型确实得到 $V = 0$ ；但它们的解释十分微妙。人们可能会认为它们都对应四面体本身的平面构型，就像被压平在底面上的帐篷，但大多数这类构型并非如此。我们可以观察到，四面体棱边方向的向量可以通过任意两个面积向量求叉积得到。对于共面面积向量构型，所有这些叉积都共线，此时四面体已经退化成一维“铅笔状”四面体，比平面构型退化程度更高。需要注意的是，尽管这些四面体是退化的，但只要对它们做微小形变就能得到非退化构型，因此在相空间中保留这些极限点是有意义的。一般而言，合并经度 $\phi = 0$ 和 $\phi = \pi$ 得到的球形形状空间上的大圆，所有满足 $V = 0$ 的构型，完全由铅笔状四面体构成。例外情况是形状空间包含平面四面体的情形，我们会在下文第三点讨论。

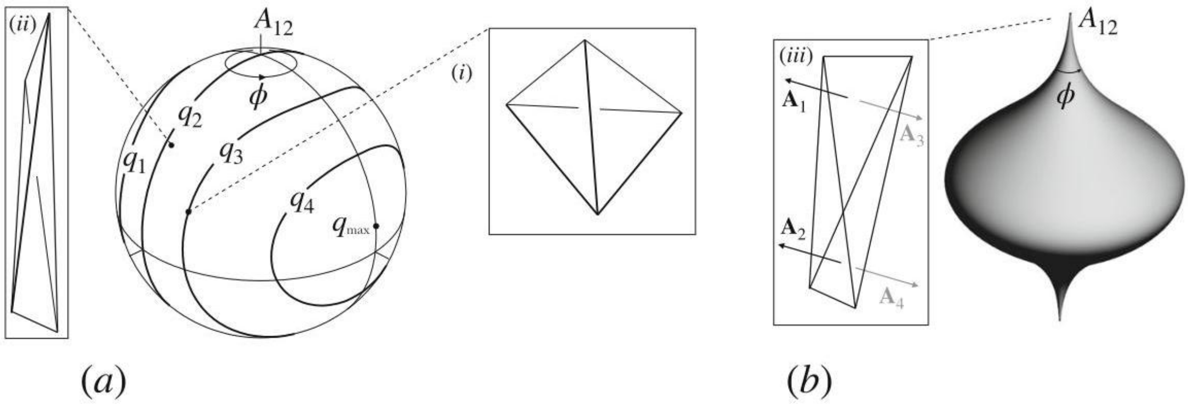


Fig. 6 (a) The phase space of shapes of a tetrahedron Φ_4 . The inset (i) shows a tetrahedron corresponding to a generic point of the space. The inset (ii) shows a tetrahedron becoming pencillike as the phase space point limits to the great circle $V = 0$. The darkened contours illustrate quantized level sets, with values q_i , of the classical volume squared $Q = V^2$. (b) When the area magnitudes are fixed to values that allow for planar tetrahedra, the space of shapes develops cusp singularities at the corresponding pole. Here the case where both $A_{12\,\min}$ and $A_{12\,\max}$ are singular is illustrated. The planar tetrahedral configuration at $A_{12\,\max}$ is shown in inset (iii)

图 6 (a) 四面体形状相空间 Φ_4 。插图 (i) 展示了对应于空间中一般点的四面体。插图 (ii) 展示了当相空间点趋近于大圆 $V = 0$ 时，四面体变成铅笔状。深色轮廓展示了经典体积平方 $Q = V^2$ 的量子水平集，其值为 q_i 。(b) 当面模量被固定为允许平面四面体存在的值时，形状空间在对应极点处形成尖点奇异。此处展示了 $A_{12\,\min}$ 和 $A_{12\,\max}$ 均为奇异的情形，插图 (iii) 展示了 $A_{12\,\max}$ 处的平面四面体构型

Second, note that the bending flow is uninterrupted by planar configurations of the area vectors and continues past them. Area vector configurations symmetrically on either side of a planar configuration can be reached by sending $\mathbf{A}_\ell \mapsto -\mathbf{A}_\ell$ and performing an overall (gauge) rotation of all the vectors. At the level of the tetrahedra, this maps outward-pointing normals into inward normals. Since the area vectors are angular momenta, this is physically the time reversal of angular momenta, and the two hemispheres on either side of the great circle $V = 0$ can be seen as time-reversed tetrahedra in this sense. For many purposes then, including the quantization below, it is sufficient to work only with the hemisphere with outward-pointing normals.

其次需要注意，弯曲流不会被面积向量的平面构型打断，会穿过这些构型继续延伸。平面构型对称两侧的面积向量构型可以通过发送 $\mathbf{A}_\ell \mapsto -\mathbf{A}_\ell$ 并对所有向量做整体 (规范) 旋转得到。在四面体层面，这一操作将外法向映射为内法向。由于面积向量就是角动量，这在物理上就是角动量的时间反演，从这个意义上说，大圆 $V = 0$ 两侧的两个半球可以看作互为时间反演的四面体。因此，对于包括下文量子化在内的许多应用场景，仅研究外法向对应的半球就足够了。

Finally, the third subtlety is that the phase space of shapes becomes singular when planar configurations of tetrahedra are possible. Planar tetrahedra are only possible if the fixed area magnitudes satisfy an additional condition: either $A_1 + A_2 = A_3 + A_4$ or $A_1 + A_2 + A_3 = A_4$ or a permutation of indices of one of these. The reason is that, for planar tetrahedra, the area vectors themselves cannot just be coplanar but must be collinear (see inset (iii) of Fig. 6). For collinear configurations of area vectors, the bending flow generated by A_{12} is singular, and the phase space of shapes becomes cusped (panel (b) of Fig. 6). The signature of these singular phase spaces is also imprinted on the quantization discussed next.

最后，第三个微妙之处在于: 当存在四面体的平面构型时，形状的相空间会变得奇异。仅当固定的面积大小满足一个附加条件时，平面四面体才可能存在: 要么满足 $A_1 + A_2 = A_3 + A_4$ ，要么满足 $A_1 + A_2 + A_3 = A_4$ ，要么是其中一种情况的指标置换。原因在于，对平面四面体来说，面积向量本身不能仅仅共面，还必须共线 (参见图 6 的内嵌图 (iii))。对于共线的面积向量构型，由 A_{12} 生成的弯曲流是奇异的，形状的相空间会形成尖点 (图 6 的 (b) 子图)。这些奇异相空间的特征也会留在接下来讨论的量子化中。

Semiclassical Quantization of Tetrahedra

四面体的半经典量子化

Quantization of the tetrahedral volume formula (79) provides a physically compelling candidate for an intertwiner to use in determining gauge-invariant states at the nodes of a 4-valent spin network. The clear difficulty with this expression is that even in the well-adapted coordinates introduced here, it is cubic in the basic variables. This means that its quantization will inherently have operator ordering ambiguities and other subtleties. While the full spectrum is not known analytically, the spin network methods described in section ”The Volume Operators” have furnished explicit expressions for the matrix elements of the corresponding volume operator $\hat{q}$ in a particular basis (see, e.g. [42], for a summary of what is known). These methods have facilitated some analytic results and numerical investigations [34-41]. A nice feature of the tetrahedral case is that the internal and external regularizations, discussed in section ”The Volume Operators”, agree in this case, and the methods presented below give a compelling argument for fixing the arbitrary constant a_0 of Eq. (44) in this case.

对四面体体积公式 (79) 的量子化，为确定四价自旋网络节点处的规范不变态提供了一个物理上合理的交叠子候选。这个表达式的明显难点在于，即使使用本文引入的适配坐标，体积在基础变量中仍是三次的。这意味着它的量子化天生存在算符排序歧义以及其他细节问题。虽然目前无法解析得到完整谱，但“体积算符”一节介绍的自旋网络方法已经给出了对应体积算符 $\hat{q}$ 在特定基下矩阵元的显式表达式 (相关研究总结参见例如文献 [42])。这些方法推动了若干解析结果与数值研究的完成 [34-41]。四面体情况的一个优良性质是，“体积算符”一节讨论的内部正则化与外部正则化在此情形下一致，而且下文介绍的方法为固定式 (44) 中的任意常数 a_0 提供了合理依据。

In this section, we will review a second, semiclassical approach to the quantization of volume. This approach has yielded several explicit analytical results and is being actively pursued to extend existing results on spatial Euclidean tetrahedra to the case of Lorentzian tetrahedra embedded in spacetime.

本节我们将综述第二种半经典体积量子化方法。该方法已经得到了多个显式解析结果，目前仍在活跃发展中，目标是将空间欧几里得四面体的已有结果推广到嵌入时空的洛伦兹四面体情形。

The cubic character of the volume is not a problem for a semiclassical approach and leads into the interesting terrain of elliptic curves. The strategy is to fix the area magnitudes $\{A_\ell\}_{\ell=1}^4$ and to study the Hamiltonian evolution generated by $Q \;:=\; V^2 = \mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)$ on the phase space Φ_4 . Taking Q as a Hamiltonian, the brackets of Eq. (72) give the evolution of any function f on Φ_4 via the flow equation $df/d\lambda = \{f, Q\}$, with λ the parameter of the volume flow, that is, $\{\lambda, Q\} = 1$.

体积的三次性质对半经典方法而言不是问题，反而将我们引向椭圆曲线这一有趣领域。该方法的策略是固定面积大小 $\{A_\ell\}_{\ell=1}^4$ ，研究相空间 Φ_4 上由 $Q \;:=\; V^2 = \mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)$ 生成的哈密顿演化。将 Q 取为哈密顿量后，式 (72) 的括号给出任意函数 f 在 Φ_4 上通过流方程 $df/d\lambda = \{f, Q\}$ 的演化，其中 λ 是体积流的参数，即 $\{\lambda, Q\} = 1$ 。

The volume evolution on Φ_4 can be expressed in a nice form in terms of the action angle coordinates $\{A_{12}, \phi\}$. Using the definition of ϕ as the angle between the two wings of the area vectors in panel (a) of Fig. 6, a cross-product calculation shows that the squared volume satisfies the relation

Φ_4 上的体积演化可以用作用角坐标 $\{A_{12}, \phi\}$ 写成简洁优美的形式。利用 ϕ 是图 6(a) 中两个面积矢量翼之间夹角的定义，叉乘计算表明体积平方满足以下关系:

$$QA_{12} = 8/9 \Delta_{12} \Delta_{34} \sin\phi, \tag{80}$$

with Δ_{12} and Δ_{34} determined by Heron’s formula

其中 Δ_{12} 和 Δ_{34} 由海伦公式确定

$$\Delta_{12} \;:=\; \frac{1}{4}\sqrt{\left[(A_1 + A_2)^2 - A_{12}^2\right]\left[A_{12}^2 - \left(A_1^2 - A_2^2\right)\right]}, \tag{81}$$

$$\Delta_{34} \;:=\; \frac{1}{4}\sqrt{\left[(A_3 + A_4)^2 - A_{12}^2\right]\left[A_{12}^2 - \left(A_3^2 - A_4^2\right)\right]}.$$

The volume evolution of A_{12}^2 follows quickly, first note

很快就能得到 A_{12}^2 的体积演化，首先注意

$$d\left(A_{12}^2\right)/d\lambda = \left\{A_{12}^2, Q\right\} = 2A_{12}\left\{A_{12}, Q\right\} = 2A_{12}\partial Q/\partial\phi, \tag{82}$$

where in the last equality the fact that A_{12} and ϕ are conjugate has been used. Using the expression (80), both to compute the partial derivative and to eliminate ϕ supplies, the central result

其中最后一步等式用到了 A_{12} 和 ϕ 共轭的性质。利用式 (80) 计算偏导并消去 ϕ 后，我们得到核心结果:

$$d\left(A_{12}^2\right)/d\lambda = 1/9\sqrt{\left(4\Delta_{12}\right)^2\left(4\Delta_{34}\right)^2 - 18A_{12}^2Q^2}. \tag{83}$$

The argument of the square root is a quartic polynomial in the squared action A_{12}^2 .

平方根的宗量是作用平方 A_{12}^2 的四次多项式。

This result demonstrates that the volume evolution is an elliptic curve. Let $x := A_{12}^2$, and $y := dx/d\lambda$; then an elliptic curve in these variables is defined by an equality between y^2 and a cubic or quartic polynomial of x . Elliptic curves are famous in physics for the role that they play in the full, non-linear integration of the simple pendulum and in rigid body dynamics. The derivation above shows that the volume evolution is integrable. Indeed, explicit expressions for $A_{12}^2(\lambda)$ in terms of Jacobi' s $\mathrm{sn}(\lambda, m)$ can be found using this equation (see [42]). The function $A_{12}^2(\lambda)$ is periodic in λ and is a parametrization of the darkened contours shown in panel (a) of Fig. 6. Of direct interest here is the fact that these curves can be used to provide a semiclassical quantization of the volume.

这一结果证明体积演化是一条椭圆曲线。设 $x := A_{12}^2$ ，且 $y := dx/d\lambda$ ；那么这些变量中的椭圆曲线由 y^2 和 x 的三次或四次多项式相等定义。椭圆曲线在物理学中十分知名，因为它在单摆的完全非线性积分和刚体动力学中发挥作用。上述推导表明体积演化是可积的。事实上，利用该方程可以得到用雅可比 $\mathrm{sn}(\lambda, m)$ 表示的 $A_{12}^2(\lambda)$ 的显式表达式 (参见文献 [42])。函数 $A_{12}^2(\lambda)$ 关于 λ 是周期函数，是图 6(a) 中深色轮廓线的参数化。此处我们直接关注的是，这些曲线可用于给出体积的半经典量子化。

The idea is to return to old quantum theory and implement the Bohr-Sommerfeld quantization rule. This rule states that the quantized values of an observable Q can be found by requiring that the action integral, call it I , computed along a quantized level set of Q , should be equal to a half-integer multiple of Planck' s constant

其核心思路是回归旧量子论，实施玻尔-索末菲量子化规则。该规则指出，可观测量 Q 的量子化取值可通过以下条件得到: 沿 Q 的量子化等值面计算的作用量积分记为 I ，其应当等于普朗克常数的半整数倍

$$I(q) = \oint_{\gamma_q} A_{12}d\phi = 2\pi(n + 1/2)\hbar, \tag{84}$$

where γ_q is the level-set contour $Q = q$ and n is a nonnegative integer, $n \in \mathbb{Z}^*$. Once the action $I(q)$ is computed, the expression is inverted to obtain the quantized values q_n of the quantized observable $\hat{q}$.

其中 γ_q 是等值面轮廓 $Q=q$，n 是非负整数 $n\in\mathbb{Z}^*$。计算得到作用量 $I(q)$ 后，反转该表达式即可得到量子化可观测量 $\hat{q}$ 的量子化取值 q_n。

This action integral can be carried out explicitly by parametrizing both A_{12} and ϕ by λ ,

该作用量积分可以通过用 λ 对 A_{12} 和 ϕ 同时参数化显式计算，

$$I(q)=\left(aK(m)-\sum_{i=1}^{4}b_i\Pi\left(\alpha_i^2,m\right)\right)q. \tag{85}$$

Here the result is expressed in terms of the complete elliptic integrals of the first kind, $K(m)$, and that of the third kind, $\Pi\left(\alpha^2,m\right)$. The parameters $\{a,b_i\}$ and the moduli $\left\{m,\alpha_i^2\right\}$, both with $i\in\{1,2,3,4\}$, are functions of the $\{A_\ell\}_{\ell=1}^4$ and of q through the roots of the quartic polynomial appearing under the square root in Eq. (83); their explicit expressions will not be needed here but can be found in [42]. Using this expression, the Bohr-Sommerfeld values for the quantized level sets q_n can be found. These Bohr-Sommerfeld values, black dots, are compared to the numerical diagonalization of the Loop Quantum Gravity spin network results, open circles, in Fig. 7. The agreement, even at lowest order in $\hbar$, is striking.

此处结果用第一类完全椭圆积分 $K(m)$ 和第三类完全椭圆积分 $\Pi\left(\alpha^2,m\right)$ 表示。参数 $\{a,b_i\}$ 和模 $\left\{m,\alpha_i^2\right\}$ (均满足 $i\in\{1,2,3,4\}$) 通过式 (83) 平方根下四次多项式的根成为 $\{A_\ell\}_{\ell=1}^4$ 和 q 的函数；本文不需要它们的显式表达式，相关结果可在文献 [42] 中找到。利用该表达式，可以得到量子化等值面 q_n 的玻尔-索末菲取值。这些玻尔-索末菲取值 (黑点) 在图 7 中与圈量子引力自旋网络的数值对角化结果 (空心圆) 进行了对比。即使在 $\hbar$ 的最低阶下，二者的一致性也十分显著。

Perturbative/Non-perturbative Connections in Quantum Tetrahedra

量子四面体中的微扰/非微扰联系

Using WKB theory, the results above can be further extended to find analytic semiclassical expressions for the volume eigenstates [130, 131]. Going beyond this, perhaps the most interesting application of these methods is currently under development [132]. This has to do with the connection between these results and the theory of quantum curves [133-137]. Above we have seen that the volume flow traces out an algebraic curve, in this case an elliptic curve. All elliptic curves can be brought to Weierstrass normal form, where the relationship between the x and y variables introduced above can be written as $y^2=4x^3-g_2x-g_3$ and the invariants $\{g_2,g_3\}$ classify the elliptic curve up to isomorphism. More generally, an algebraic curve is given by the vanishing of a polynomial in x and $y, P(x,y)=0$. A quantum curve is a quantization of this relation where x and y are promoted to operators in the standard manner

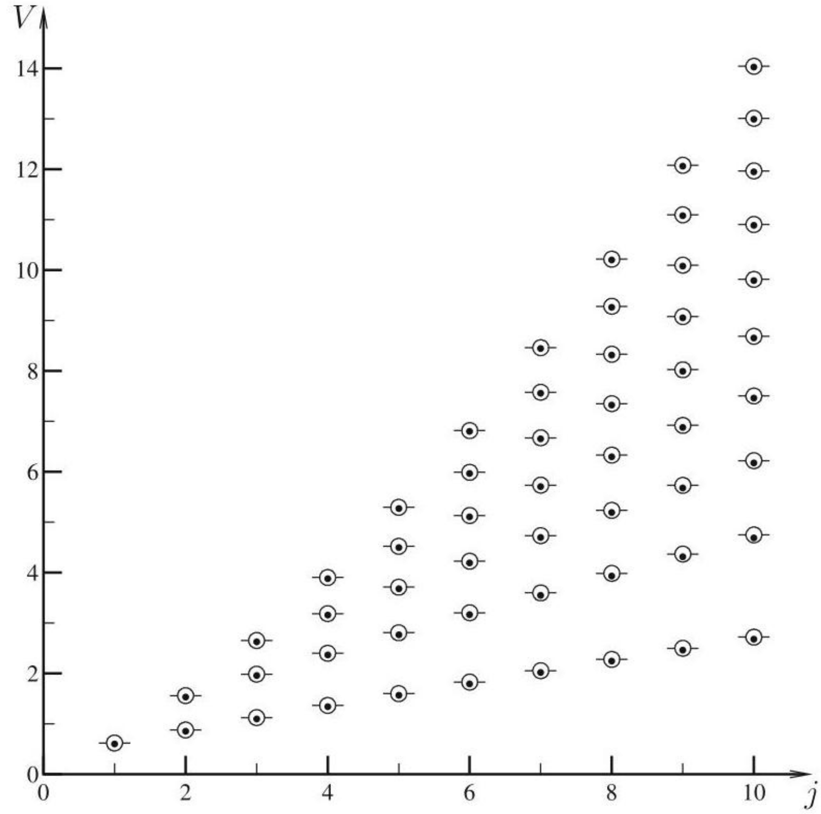
利用 WKB 理论，我们可以进一步拓展上述结果，得到体积本征态的解析半经典表达式 [130, 131]。除此之外，这些方法目前最受关注的应用方向仍在开发中 [132]，该应用与上述结果和量子曲线理论 [133-137] 之间的联系有关。前文我们已经看到，体积流描绘出一条代数曲线，在我们讨论的情形中这是一条椭圆曲线。所有椭圆曲线都可以化为魏尔斯特拉斯标准型，其中前文引入的变量 x 和 y 之间的关系可以写为 $y^2 = 4x^3 - g_2 x - g_3$ ，不变量 $\{g_2, g_3\}$ 对椭圆曲线进行同构分类。更一般地说，一条代数曲线由 x 和 $y, P(x, y) = 0$ 中多项式的零点给出。量子曲线是对该关系的量子化，即将 x 和 y 按标准方式提升为算符

$$\hat{x} = x,\ \hat{y} = \frac{\hbar}{i}\frac{d}{dx}, \tag{86}$$

and $P(x, y)$ is promoted to an operator $\widehat{P}(\hat{x}, \hat{y}; \hbar)$. The operator $\widehat{P}$ is a differential operator in x , with coefficients that are polynomials in x , and, importantly, possibly power series in $\hbar$. States $\psi(x)$ annihilated by such an operator, $\widehat{P}(\hat{x}, \hat{y}; \hbar)\psi = 0$, have turned out to be of central importance in knot theory, Chern-Simons theories, and matrix models.

，且 $P(x, y)$ 被提升为算符 $\widehat{P}(\hat{x}, \hat{y}; \hbar)$ 。算符 $\widehat{P}$ 是 x 上的微分算符，其系数是 x 上的多项式，重要的是，它还可以是 $\hbar$ 上的幂级数。被该算符 $\widehat{P}(\hat{x}, \hat{y}; \hbar)\psi = 0$ 湮灭的量子态 $\psi(x)$ 已被证明在纽结理论、陈-西蒙斯理论和矩阵模型中处于核心地位。

a



b

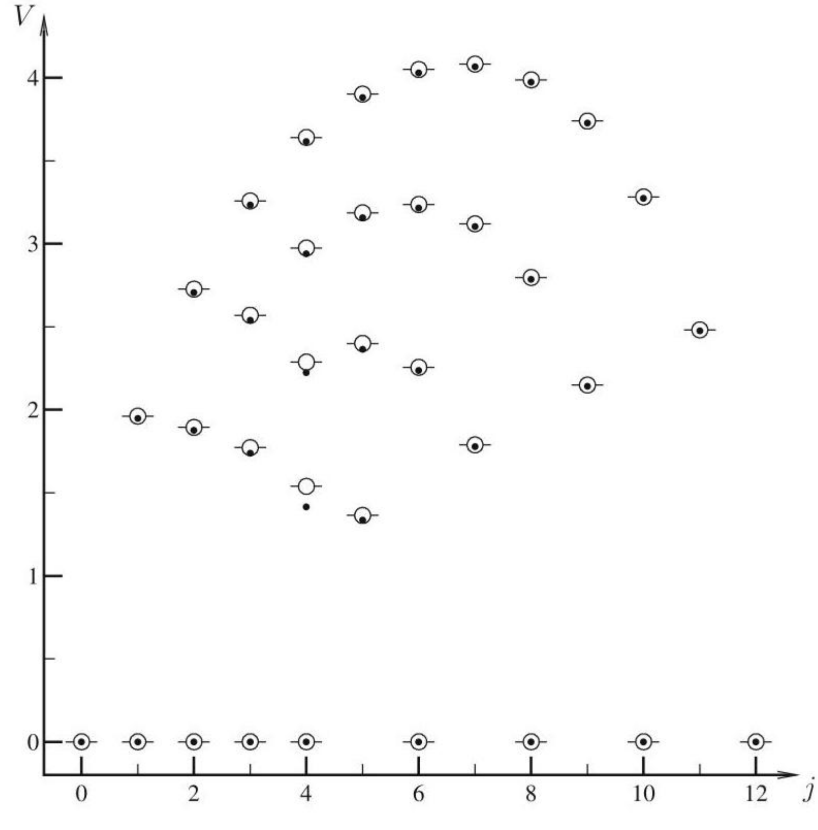


Fig. 7 Comparison of the Bohr-Sommerfeld and Loop Quantum Gravity volume spectra. (a) At left: a tetrahedron with area eigenvalues determined by $\{j, j, j, j+1\}$. (b) At right: a tetrahedron with spins $\{4, 4, 4, j\}$ and j varying in its allowed range. The Bohr-Sommerfeld values of the volume are represented by black dots and the eigenvalues of the loop-gravity volume operator as open circles. Recall the spins j and areas are related by $A_\ell = \sqrt{j_\ell (j_\ell + 1)}\hbar\kappa\gamma$; for these plots $\hbar\kappa\gamma = 1$

图 7 玻尔-索末菲体积谱与圈量子引力体积谱的对比。(a) 左侧: 面积本征值由 $\{j, j, j, j+1\}$ 确定的四面体。(b) 右侧: 自旋 $\{4, 4, 4, j\}$ 和 j 在允许范围内变化的四面体。体积的玻尔-索末菲值用黑点表示, 圈量子引力体积算符的本征值用空心圆表示。回顾可知，自旋 j 和面积满足关系 $A_\ell = \sqrt{j_\ell (j_\ell + 1)}\hbar\kappa\gamma$ ；这些图中 $\hbar\kappa\gamma = 1$

The case of elliptic curves is particularly interesting. In this case, Basar, Dunne, and Ünsal have explained, in [138], an intriguing perturbative/non-perturbative connection using WKB theory. For concreteness, consider an energy eigenstate of an anharmonic quartic oscillator. At the classical level, a particle localized in the left well of the potential has an energy exactly degenerate with one localized in the right well (see panel (a) of Fig. 8). However, quantum mechanically, this degeneracy is broken by non-perturbative tunneling between the two wells. The energy eigenstates have an exponentially small splitting ΔE , shown schematically in the diagram by the dot-dashed curves. Surprisingly, the quantization of the energy level E_n and the splitting between this level and its nearest neighbor, ΔE_n , are not independent. The connection between these two quantities is explained by viewing the Hamiltonian level set $H = E$ as the elliptic curve $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$.

椭圆曲线的情形格外有趣。在该情形下，巴萨尔 (Basar)、邓恩 (Dunne) 和温萨尔 (Ünsal) 在文献 [138] 中阐述了利用 WKB 理论得到的有趣微扰/非微扰联系。为具体说明，我们考虑非调和四次谐振子的能量本征态。在经典层面，局域在势阱左阱的粒子和局域在右阱的粒子能量完全简并 (参见图 8 的面板 (a))。但在量子力学中，这种简并会被两个阱之间的非微扰隧穿效应打破。能量本征态存在指数小的劈裂 ΔE ，点划线在图中示意性标出了这一劈裂。令人惊讶的是，能级 E_n 的量子化和该能级与其最近邻能级之间的劈裂 ΔE_n 并非独立的。通过将哈密顿能级集合 $H = E$ 视为椭圆曲线 $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$ ，可以解释这两个量之间的联系。

As above, the Bohr-Sommerfeld quantization of the energy is computed by fixing the level set $H = E$ of the Hamiltonian and computing the action over a full period between, say, the two left-most turning points of the potential. Also interesting is the action integral connecting the two central turning points of the potential, which turns out to be what controls the level splitting ΔE_n [139, 140] . The character of these two actions emerges more clearly if one complexifies the variables x and p . Then, the elliptic curve $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$ is the genus-one Riemann surface depicted in the second panel of Fig. 8. The two action integrals are simply the integrals of the action one-form, pdx , along the independent a - and b -cycles of this torus. Going back to Riemann, the two periods, given by the energy derivative of these actions, were recognized to be related. In fact, these two periods are two independent solutions to what is known as a Picard-Fuchs equation. Basar, Dunne, and Unsal show that this relation extends to each order in $\hbar$ [138]. This means that if you know the Bohr-Sommerfeld energy levels E_n to a given order in $\hbar$, you can actually algorithmically construct their non-perturbative splitting ΔE_n to the same order in $\hbar$.

同上，玻尔-索末菲能量量子化的计算方法是: 固定哈密顿量的水平集 $H = E$，计算势能两个最左回转点之间一个完整周期的作用量。同样值得关注的是连接势能两个中心回转点的作用积分，它正是控制能级分裂 ΔE_n [139, 140] 的量。如果将变量 x 和 p 复化，这两个作用量的性质会变得更加清晰。此时，椭圆曲线 $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$ 就是图 8 第二幅面板中所示的亏格 1 黎曼曲面。这两个作用积分恰好就是作用 1-形式 pdx 沿该环面两个独立的 a -循环和 b -循环的积分。追溯到黎曼的工作，人们早已认识到，由这些作用量对能量的导数给出的两个周期是相互关联的。实际上，这两个周期正是著名的皮卡-富克斯方程的两个独立解。巴萨尔、邓恩和乌内萨尔证明，该关系可以推广到 $\hbar$ 的每一阶 [138]。这意味着，如果已知玻尔-索末菲能级 E_n 在 $\hbar$ 展开到给定阶的结果，你就可以通过算法构造出非微扰分裂 ΔE_n 在 $\hbar$ 下相同阶的结果。

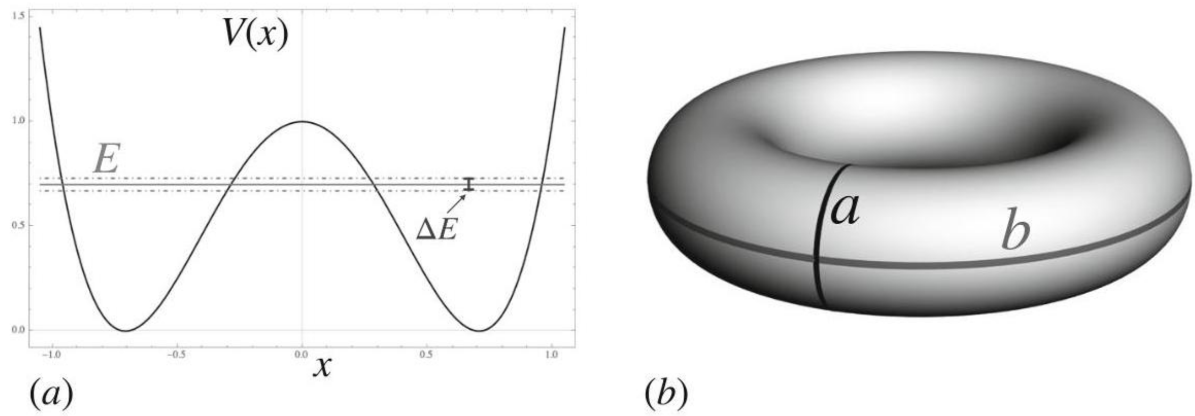


Fig. 8 (a) The potential energy $V(x)$ of a symmetric quartic oscillator with Hamiltonian $H(x, p) = p^2/2m + \left(2x^2 - 1\right)^2$. The level set $H = E$ is depicted by a solid gray curve. The quantum level splitting between the symmetric and antisymmetric energy eigenstates is illustrated as the gap ΔE between the dot-dashed horizontal lines. (b) The elliptic curves $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$ for the quartic oscillator and (83) for the quantum tetrahedron can both be seen as genus-one Riemann surfaces. The action integrals along the topologically distinct a - and b -cycles are needed to compute the semiclassical approximations to the level quantization and non-perturbative tunneling of the quantum systems

图 8 (a) 对称四次谐振子的势能 $V(x)$，其哈密顿量为 $H(x, p) = p^2/2m + \left(2x^2 - 1\right)^2$。灰色实线绘制了水平集 $H = E$。对称与反对称能量本征态之间的量子能级分裂表示为点划线水平线之间的间隙 ΔE。(b) 四次谐振子的椭圆曲线 $p^2 = 2m\left[E - \left(2x^2 - 1\right)^2\right]$ 和量子四面体的方程 (83) 都可视为亏格 1 黎曼曲面。沿拓扑不同的 a -循环和 b -循环的作用积分是计算量子系统能级量子化和非微扰隧穿的半经典近似所必需的。

Many of the same mathematical structures are present in the volume quantization of the tetrahedron reviewed above. The elliptic curve corresponding to the volume (Eq. (83)) can also be viewed as a complex, genus-one Riemann surface. Recent work has uncovered the Picard-Fuchs equation for the periods of this family of Riemann surfaces, which are parameterized by the volume of the tetrahedron [131, 132]. This work has established that one of these periods, the one along the a - cycle, is canonically associated with a real section of these Riemann surfaces; this period is the one associated with the action computed along the $Q = q$ contour that was used above to find the Bohr-Sommerfeld volume eigenvalues. Even more intriguingly, these researchers have shown that the second period is associated with a purely imaginary contour, the b -

cycle, that can be put in correspondence with three-dimensional Lorentzian tetrahedra. These are tetrahedra embedded in a three-dimensional Minkowski space of signature $(-,+,+)$. These tetrahedra still satisfy a Minkowski theorem, but their squared volumes are negative. This work opens the way to a geometric and analytic understanding of non-perturbative tunneling between grains of space and grains of spacetime.

前文回顾的四面体体积量子化也存在许多相同的数学结构。对应体积的椭圆曲线 (式 (83)) 同样可以视为复亏格 1 黎曼曲面。最近的工作得到了由四面体体积参数化的该族黎曼曲面周期所满足的皮卡-富克斯方程 [131, 132]。该研究证实，其中一个沿 a -循环的周期，正则对应于这些黎曼曲面的实截面；这个周期对应于前文用来求解玻尔-索末菲体积本征值的 $Q=q$ 围道上计算得到的作用量。更引人关注的是，这些研究者还证明，第二个周期对应纯虚围道，即 b -循环，它可以与三维洛伦兹四面体建立对应关系。这类四面体嵌入号差为 $(-,+,+)$ 的三维闵可夫斯基空间中，仍然满足闵可夫斯基定理，但平方体积为负。这项工作为从几何和解析角度理解空间晶粒与时空晶粒之间的非微扰隧穿开辟了道路。

Discrete Geometry Wrap-Up

离散几何总结

This section highlighted the emergence of Riemannian quantum geometry from its deep classical roots in Euclidean geometry. The same striking insight that Einstein brought to spacetime applies to discrete polyhedral geometries: they can be made into dynamical systems whose geometrical shapes evolve. Remarkably, these geometrical shapes can be quantized and have a clear correspondence with the spin network states of loop quantum gravity. This section showed how the states associated with the gauge-invariant nodes of a spin network can be viewed as quantum polyhedra and explored the case of the quantum tetrahedron in detail. Limitations of space precluded the treatment of other geometrical operators, but the discrete geometric approach also lends insight into the other operators introduced in section ”Quantum Geometry”.

本节重点介绍了黎曼量子几何如何从其欧几里得几何的深厚古典根基中涌现。爱因斯坦对时空提出的那个惊人洞见同样适用于离散多面体几何: 它们可以构成几何形状会演化的动力学系统。值得注意的是，这些几何形状可以被量子化，并且与圈量子引力的自旋网络态有明确的对应关系。本节说明了自旋网络规范不变节点对应的态如何可以被看作量子多面体，并详细探讨了量子四面体的情况。受篇幅限制，我们无法讨论其他几何算符，但离散几何方法也能为“量子几何”一节中介绍的其他算符提供洞见。

In particular, it was demonstrated how the area and volume of a quantum tetrahedron, which corresponds to a 4-valent node of a spin network, attain discrete spectra in a semiclassical approach to the quantization. Indeed, these spectra give a complete set of quantum numbers for the 4-valent node. Higher-valence nodes are tricky in more than one respect. For the 5-valent case, the polyhedral picture still supplies a clear proposal for a volume Hamiltonian [141], but the dynamics generated by this Hamiltonian is chaotic [141, 142]. Even with more complicated volume spectra in hand, there is the issue that with higher-valence nodes, the dimension of the space of intertwiners grows, and further quantum labels are needed. The action angle variables introduced above always provide a complete set of coordinates of the phase space, and there is a corresponding set of quantum numbers, the angular momentum recoupling channels [143]. While the recoupling channels are often used, another approach has also been developed that uses an enlarged Hilbert space where only the total bounding area of the quantum grain of space is fixed, and this is termed the U (N) framework [119,144-146]. More general approaches looking for a full set of conserved charges at the boundary of spacetime are also under development [147-150].

> 具体而言，我们证明了对应自旋网络 4 价节点的量子四面体，其面积和体积在半经典量子化方法中获得离散谱。实际上，这些谱构成了 4 价节点的完整量子数集合。高价节点在多个层面都处理起来十分棘手。对于 5 价节点，多面体图像仍然可以给出体积哈密顿量的明确方案 [141]，但该哈密顿量生成的动力学是混沌的 [141, 142]。即便我们得到了更复杂的体积谱，高价节点仍存在一个问题: 交缠子空间的维数会增长，因此需要额外的量子标记。前文介绍的作用角变量总能提供相空间的一整套坐标，与之对应的是一组量子数，即角动量重耦通道 [143]。除了常用的重耦通道，学界也发展出了另一种方法: 使用扩大的希尔伯特空间，仅固定空间量子 grain 的总边界面积，这个框架被称为 U (N) 框架 [119,144-146]。目前也有更一般的方法在发展中，这类方法旨在寻找时空边界上全套守恒荷 [147-150]。

There were many things that we could not cover in this section. One of the reasons to explain that the phase space develops singularities when planar tetrahedra arise (see Fig. 6) is that these are closely related to the role of asymptotic resurgence [151-153] in the study of quantum tetrahedra. Planar configurations show up as logarithmic singularities of the elliptic functions of the third kind and contribute to the zero-mode terms of the resurgent transseries [132]. Another fascinating direction to go in is that of constant-curvature tetrahedra and 4-simplices. In three spacetime dimensions, constant curvature tetrahedra quickly lead into the rich mathematical terrain of Poisson-Lie groups and quantum groups [154-157]. The four-dimensional case of constant curvature simplices is deeply connected with the algebraic and quantum curves discussed above [158-161]. This work reawakened the longstanding exchange between Chern-Simons theory and loop gravity [162-164], in particular studying connections to complex Chern-Simons theory [165-167]. The work [158] pioneered the study of graph complement manifolds, generalizing the extensive work on knot complements [168].

> 本节我们还有很多内容未能涵盖。当出现平面四面体时相空间会产生奇点 (见图 6)，对该现象的一种解释是，这和量子四面体研究中的渐近复活渐近作用密切相关 [151-153]。平面构型会表现为第三类椭圆函数的对数奇点，对复活跨级数的零模项有贡献 [132]。另一个引人入胜的研究方向是常曲率四面体和 4 单形。在三维时空下，常曲率四面体很快会进入泊松-李群和量子群丰富的数学领域 [154-157]。四维情况下，常曲率单形和前文讨论的代数曲线与量子曲线深度关联 [158-161]。这项工作重新唤醒了陈-西蒙斯理论和圈引力之间存在已久的交流 [162-164]，特别是对复陈-西蒙斯理论关联的研究 [165-167]。文献 [158] 开创了图补流形的研究，推广了关于纽结补的大量已有工作 [168]。

Throughout this chapter, we have largely focused on spatial geometry. However, the discrete geometry approach does not stop there and can also be used to build up spacetime. This leads into the path integral approach of spin foam models [169- 172], the subject of Chap. 86, ”Spin Foams: Foundations” of this handbook. Indeed, viewing spin foam models as the gluing of discrete geometries has been productive and recently led to attempts to simplify the structure of these path integrals in models known as effective spin foams [173-175].

> 在整章中，我们主要聚焦于空间几何。但离散几何方法并不止步于此，它也可以用来构建时空。这就引出了自旋泡沫模型的路径积分方法 [169-172]，这也是本手册第 86 章“自旋泡沫: 基础”的主题。实际上，将自旋泡沫模型看作离散几何的粘合是富有成效的，最近这一思路推动了对这类路径积分结构的简化尝试，得到的模型被称为有效自旋泡沫 [173-175]。

Conclusion

结论

We offer a brief conclusion focused on drawing connections to other parts of this handbook. The emergence of Riemannian quantum geometry informs and is informed by the developments discussed throughout this handbook. The discreteness of quantum geometry has posed intriguing questions about spacetime entanglement, leading to a fruitful exchange with quantum information, the topic of Chap. 95, ”Loop Quantum Gravity and Quantum Information,” and to a long history of ideas in black hole physics, the subject of - Chaps. 91, ”Black Hole Entropy in Loop Quantum Gravity” and - 92, ”Quantum Geometry and Black Holes”. While the focus of this chapter was on spatial geometries, the rich set of ideas around the Hamiltonian dynamics of these geometries is covered in Chaps. 84, ”Hamiltonian Theory: Dynamics” and 85, ”Hamiltonian Theory: Generalizations to Higher Dimensions, Supersymmetry, and Modified Gravity,” and the development of a path integral approach, in spin foam models, is covered in Chap. 86, ”Spin Foams: Foundations”. The computational methods necessary to calculate the amplitudes of a spin foam are covered in Chap. 87, ”Spinfoams and High-Performance Computing”. Enrichments of the notion of a spin network are covered in Chap. 88, ”Graphical Calculus of Spin Networks” on graphical calculus and one on the Chapter Boundary Degrees of Freedom in Loop Quantum Gravity. Applications to cosmology are discussed in Chaps. 89, ”Loop Quantum Cosmology: Physics of Singularity Resolution and Its Implications” and -90, ”Loop Quantum Cosmology: Relation Between Theory and Observations”, and the essential topic of the continuum limit of loop quantum gravity is taken up in Chap. 93, ”Spin Foams, Refinement Limit, and Renormalization”. Finally, the volume concludes on the philosophical foundations of the theory in - Chap. 96, ”Philosophical Foundations of Loop Quantum Gravity.”

我们给出一个简要结论，重点是建立与本手册其他部分的关联。黎曼量子几何的出现，既为本手册各处讨论的研究进展提供参考，也从这些进展中获取支撑。量子几何的离散性对时空纠缠提出了诸多引人深思的问题，推动了与量子信息领域的富有成果的交流——这是第 95 章《圈量子引力与量子信息》的主题，同时也衍生出黑洞物理学中源远流长的诸多思想，相关内容见第 91 章《圈量子引力中的黑洞熵》和第 92 章《量子几何与黑洞》。本章的研究核心是空间几何，而关于这些几何的哈密顿动力学的丰富思想，收录在第 84 章《哈密顿理论: 动力学》和第 85 章《哈密顿理论: 向高维、超对称与修正引力的推广》；自旋泡沫模型中的路径积分方法的发展，收录在第 86 章《自旋泡沫: 基础》。计算自旋泡沫振幅所需的计算方法，收录在第 87 章《自旋泡沫与高性能计算》。自旋网络概念的拓展，一部分收录在讲解图形演算的第 88 章《自旋网络的图形演算》，另一部分收录在《圈量子引力中的边界自由度》一章。宇宙学应用的相关讨论见第 89 章《圈量子宇宙学: 奇点消解的物理及其启示》和第 90 章《圈量子宇宙学: 理论与观测的关联》，圈量子引力连续极限这一核心主题，在第 93 章《自旋泡沫、粗化极限与重整化》中展开讨论。最后，本卷以对该理论哲学基础的讨论收尾，相关内容见第 96 章《圈量子引力的哲学基础》。

Acknowledgments H. M. H. was supported by grant no. 62312 from the John Templeton Foundation, as part of the ’The Quantum Information Structure of Spacetime’ Project (QISS) and through the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities.

致谢 H. M. H. 的研究得到了约翰·邓普顿基金会编号 62312 资助项目的支持，该项目属于“时空的量子信息结构”项目 (QISS)，同时得到了圆周理论物理研究所的资助。圆周理论物理研究所的研究部分得到了加拿大政府通过加拿大创新、科学与经济发展部，以及安大略省通过大学与学院厅的支持。

JL was supported by grants of the Polish Narodowe Centrum Nauki number 2018/29/B/ST2/ 01250 and number 2018/30/Q/ST2/00811.

JL 得到了波兰国家科学中心编号 2018/29/B/ST2/01250 与编号 2018/30/Q/ST2/00811 项目的资助。

H. S. acknowledges the contribution of the COST Action CA18108.

H. S. 感谢 COST 行动 CA18108 提供的支持。

References

参考文献

1. A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, J. Math. Phys. 36, 6456 (1995). https://doi.org/10.1063/1.531252

2. T. Thiemann, Modern canonical quantum general relativity (2001). arXiv preprint gr-qc/0110034

3. A. Ashtekar, J. Lewandowski, Class. Quant. Grav. 21, R53 (2004). https://doi.org/10.1088/ 0264-9381/21/15/R01

4. T. Thiemann, Modern Canonical Quantum General Relativity. Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2007). https://doi.org/10.1017/ CBO9780511755682

5. C. Rovelli, Living Rev. Rel. 11, 5 (2008)

6. S. Holst, Phys. Rev. D 53, 5966 (1996). https://doi.org/10.1103/PhysRevD.53.5966

7. J. Lee, R.M. Wald, J. Math. Phys. 31, 725 (1990). https://doi.org/10.1063/1.528801

8. A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986). https://doi.org/10.1103/PhysRevLett.57.2244

9. J.F. Barbero G., Phys. Rev. D 51, 5507 (1995). https://doi.org/10.1103/PhysRevD.51.5507

10. A. Ashtekar, J. Lewandowski, Class. Quant. Grav. 14, A55 (1997). https://doi.org/10.1088/ 0264-9381/14/1A/006

11. J. Lewandowski, A. Okolow, H. Sahlmann, T. Thiemann, Commun. Math. Phys. 267, 703 (2006). https://doi.org/10.1007/s00220-006-0100-7

12. A. Ashtekar, A. Corichi, J.A. Zapata, Class. Quant. Grav. 15, 2955 (1998). https://doi.org/10.1088/0264-9381/15/10/006

13. A.S. Cattaneo, A. Perez, Class. Quant. Grav. 34(10), 107001 (2017). https://doi.org/10.1088/ 1361-6382/aa69b4

14. A. Ashtekar, J. Lewandowski, J. Math. Phys. 36, 2170 (1995). https://doi.org/10.1063/1.531037

15. T. Thiemann, Class. Quant. Grav. 18, 3293 (2001). https://doi.org/10.1088/0264-9381/18/17/ 301

16. A. Stottmeister, T. Thiemann, arXiv preprint arXiv:1312.3657 (2013)

17. T. Koslowski, H. Sahlmann, SIGMA 8, 026 (2012). https://doi.org/10.3842/SIGMA.2012.026

18. A. Ashtekar, J. Lewandowski, J. Geom. Phys. 17, 191 (1995). https://doi.org/10.1016/0393- 0440(95)00028-G

19. C. Fleischhack, Commun. Math. Phys. 285, 67 (2009). https://doi.org/10.1007/s00220-008- 0593-3

20. J.C. Baez, Adv. Math. 117, 253 (1996). https://doi.org/10.1006/aima.1996.0012

21. S. Lojasiewicz, Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matem-atiche Ser. 3, 18(4), 449 (1964)

22. E. Bierstone, P.D. Milman, Publications Mathématiques de l'IHÉS 67, 5 (1988)

23. K. Giesel, T. Thiemann, Class. Quant. Grav. 24, 2465 (2007). https://doi.org/10.1088/0264- 9381/24/10/003

24. K. Giesel, T. Thiemann, Class. Quant. Grav. 24, 2499 (2007). https://doi.org/10.1088/0264- 9381/24/10/004

25. K. Giesel, T. Thiemann, Class. Quant. Grav. 24, 2565 (2007). https://doi.org/10.1088/0264- 9381/24/10/005

26. K. Giesel, T. Thiemann, Class. Quant. Grav. 27, 175009 (2010). https://doi.org/10.1088/0264-9381/27/17/175009

27. N. Grot, C. Rovelli, J. Math. Phys. 37, 3014 (1996). https://doi.org/10.1063/1.531527

28. C. Rovelli, L. Smolin, Nucl. Phys. B 442, 593 (1995). https://doi.org/10.1016/0550- 3213(95)00150-Q. [Erratum: Nucl. Phys. B 456, 753-754 (1995)]

29. C. Rovelli, Phys. Rev. D 47, 1703 (1993). https://doi.org/10.1103/PhysRevD.47.1703.[Erratum: Phys. Rev. D 87, 089902 (2013)]

30. G.J. Fernando Barbero, J. Lewandowski, E.J.S. Villasenor, Phys. Rev. D 80, 044016 (2009). https://doi.org/10.1103/Phys

31. S.A. Major, Class. Quant. Grav. 16, 3859 (1999). https://doi.org/10.1088/0264-9381/16/12/ 307

32. J. Lewandowski, Class. Quant. Grav. 14, 71 (1997). https://doi.org/10.1088/0264-9381/14/1/ 010

33. A. Ashtekar, J. Lewandowski, Adv. Theor. Math. Phys. 1, 388 (1998). https://doi.org/10.4310/ ATMP.1997.v1.n2.a8

34. E. Bianchi, H.M. Haggard, Phys. Rev. Lett. 107, 011301 (2011). https://doi.org/10.1103/ Phys-RevLett.107.011301

35. J. Brunnemann, D. Rideout, Class. Quant. Grav. 25, 065001 (2008). https://doi.org/10.1088/ 0264-9381/25/6/065001

36. J. Brunnemann, D. Rideout, Class. Quant. Grav. 27, 205008 (2010). https://doi.org/10.1088/ 0264-9381/27/20/205008

37. T. Thiemann, J. Math. Phys. 39, 3347 (1998). https://doi.org/10.1063/1.532259

38. J. Brunnemann, T. Thiemann, Class. Quant. Grav. 23, 1289 (2006). https://doi.org/10.1088/ 0264-9381/23/4/014

39. K. Giesel, T. Thiemann, Class. Quant. Grav. 23, 5667 (2006). https://doi.org/10.1088/0264- 9381/23/18/011

40. K. Giesel, T. Thiemann, Class. Quant. Grav. 23, 5693 (2006). https://doi.org/10.1088/0264- 9381/23/18/012

41. J. Schliemann, Class. Quant. Grav. 30, 235018 (2013). https://doi.org/10.1088/0264-9381/ 30/23/235018

42. E. Bianchi, H.M. Haggard, Phys. Rev. D 86, 124010 (2012). https://doi.org/10.1103/ PhysRevD.86.124010

43. R. Penrose, in Quantum Theory and Beyond, ed. by T. Bastin (Cambridge University Press, Cambridge, 1971), pp. 151-180

44. R. Penrose, in Magic Without Magic, ed. by J. Klauder (Freeman, San Francisco, 1972), pp. 333-354

45. E. Bianchi, P. Doná, S. Speziale, Phys. Rev. D 83, 044035 (2011). https://doi.org/10.1103/ Phys-RevD.83.044035

46. Y. Ma, Y. Ling, Phys. Rev. D 62, 104021 (2000). https://doi.org/10.1103/PhysRevD.62.104021

47. J. Lewandowski, H. Sahlmann, Phys. Rev. D 93(2), 024042 (2016). https://doi.org/10.1103/ Phys-RevD.93.024042

48. T. Thiemann, Class. Quant. Grav. 15, 839 (1998). https://doi.org/10.1088/0264-9381/15/4/ 011

49. E. Bianchi, Nucl. Phys. B 807(3), 591 (2009). https://doi.org/10.1016/j.nuclphysb.2008.08.013.https://www.sciencedire

50. T. Thiemann, J. Math. Phys. 39, 3372 (1998). https://doi.org/10.1063/1.532445 51. T.E. Regge, Il Nuovo Cimento (1955-1965) 19, 558 (1961)

52. E. Alesci, M. Assanioussi, J. Lewandowski, Phys. Rev. D 89(12), 124017 (2014). https://doi.org/10.1103/PhysRevD.89.1
53. E.R. Livine, S. Speziale, Phys. Rev. D 76, 084028 (2007). https://doi.org/10.1103/PhysRevD.76.084028
54. L. Freidel, S. Speziale, Phys. Rev. D 82, 084040 (2010). https://doi.org/10.1103/PhysRevD.82.084040
55. T. Thiemann, Class. Quant. Grav. 18, 2025 (2001). https://doi.org/10.1088/0264-9381/18/11/ 304
56. T. Thiemann, O. Winkler, Class. Quant. Grav. 18, 4629 (2001). https://doi.org/10.1088/0264-9381/18/21/315
57. T. Thiemann, O. Winkler, Class. Quant. Grav. 18, 4997 (2001). https://doi.org/10.1088/0264-9381/18/23/302
58. T. Thiemann, O. Winkler, Class. Quant. Grav. 18, 2561 (2001). https://doi.org/10.1088/0264-9381/18/14/301
59. B. Bahr, T. Thiemann, Class. Quant. Grav. 26, 045011 (2009). https://doi.org/10.1088/0264- 9381/26/4/045011
60. A. Zipfel, T. Thiemann, Phys. Rev. D 93(8), 084030 (2016). https://doi.org/10.1103/ PhysRevD.93.084030
61. B.C. Hall, J. Funct. Anal. 122(1), 103 (1994)
62. B.C. Hall, Coherent States and Their Applications (2018), pp. 41-61
63. B. Bahr, T. Thiemann, Class. Quant. Grav. 26, 045012 (2009). https://doi.org/10.1088/0264- 9381/26/4/045012
64. H. Sahlmann, T. Thiemann, O. Winkler, Nucl. Phys. B 606, 401 (2001). https://doi.org/10.1016/S0550-3213(01)00226-7
65. T. Thiemann, O. Winkler, Class. Quant. Grav. 18(23), 4997 (2001)
66. T. Thiemann, Class. Quant. Grav. 23, 2063 (2006). https://doi.org/10.1088/0264-9381/23/6/ 013
67. E. Alesci, F. Cianfrani, Phys. Rev. D 87(8), 083521 (2013). https://doi.org/10.1103/ PhysRevD.87.083521
68. E. Alesci, F. Cianfrani, EPL 111(4), 40002 (2015). https://doi.org/10.1209/0295-5075/111/ 40002
69. I. Mäkinen, Phys. Rev. D 102(10), 106010 (2020). https://doi.org/10.1103/PhysRevD.102.106010
70. J. Lewandowski, I. Mäkinen, Phys. Rev. D 106(4), 046013 (2022). https://doi.org/10.1103/ PhysRevD.106.046013
71. J. Lewandowski, I. Mäkinen, Scalar curvature operator for quantum-reduced loop gravity (2022). arXiv preprint 2211.04826
72. A. Corichi, K.V. Krasnov, Mod. Phys. Lett. A 13, 1339 (1998). https://doi.org/10.1142/ S0217732398001406
73. T. Thiemann, Class. Quant. Grav. 15, 1487 (1998). https://doi.org/10.1088/0264-9381/15/6/ 006
74. H.A. Morales-Tecotl, C. Rovelli, Phys. Rev. Lett. 72, 3642 (1994). https://doi.org/10.1103/ PhysRevLett.72.3642
75. J.C. Baez, K.V. Krasnov, J. Math. Phys. 39, 1251 (1998). https://doi.org/10.1063/1.532400
76. M. Bojowald, R. Das, Phys. Rev. D 78, 064009 (2008). https://doi.org/10.1103/PhysRevD.78.064009
77. R. Mansuroglu, H. Sahlmann, Phys. Rev. D 103(6), 066016 (2021). https://doi.org/10.1103/ PhysRevD.103.066016
78. R. Gambini, O. Obregon, J. Pullin, Nucl. Phys. B 460, 615 (1996). https://doi.org/10.1016/ 0550-3213(95)00582-X
79. Y. Ling, L. Smolin, Phys. Rev. D 61, 044008 (2000). https://doi.org/10.1103/PhysRevD.61.044008
80. N. Bodendorfer, T. Thiemann, A. Thurn, Phys. Lett. B 711, 205 (2012). https://doi.org/10.1016/j.physletb.2012.04.003
81. N. Bodendorfer, T. Thiemann, A. Thurn, Class. Quant. Grav. 30, 045006 (2013). https://doi.org/10.1088/0264-9381/30/4/045006
82. N. Bodendorfer, T. Thiemann, A. Thurn, Class. Quant. Grav. 30, 045007 (2013). https://doi.org/10.1088/0264-9381/30/4/045007
83. K. Eder, Super Cartan geometry and the super Ashtekar connection (2020). arXiv preprint 2010.09630
84. K. Eder, H. Sahlmann, JHEP 07, 071 (2021). https://doi.org/10.1007/JHEP07(2021)071

85. M. Campiglia, M. Varadarajan, Class. Quant. Grav. 31, 075002 (2014). https://doi.org/10.1088/0264-9381/31/7/075002

86. M. Campiglia, M. Varadarajan, Class. Quant. Grav. 31, 175009 (2014). https://doi.org/10.1088/0264-9381/31/17/175009

87. T.A. Koslowski, Dynamical Quantum Geometry (DQG Programme) (2007). arXiv preprint 0709.3465

88. H. Sahlmann, Class. Quant. Grav. 27, 225007 (2010). https://doi.org/10.1088/0264-9381/27/ 22/225007

89. H. Sahlmann, R. Seeger, Phys. Rev. D 101(10), 106018 (2020). https://doi.org/10.1103/ PhysRevD.101.106018

90. M. Varadarajan, Class. Quant. Grav. 30, 175017 (2013). https://doi.org/10.1088/0264-9381/ 30/17/175017

91. R. Gambini, J. Griego, J. Pullin, Phys. Lett. B 413, 260 (1997). https://doi.org/10.1016/ S0370-2693(97)01048-4

92. B. Dittrich, M. Geiller, Class. Quant. Grav. 32(13), 135016 (2015). https://doi.org/10.1088/ 0264-9381/32/13/135016

93. B. Dittrich, M. Geiller, Class. Quant. Grav. 32(11), 112001 (2015). https://doi.org/10.1088/ 0264-9381/32/11/112001

94. B. Bahr, B. Dittrich, M. Geiller, Class. Quant. Grav. 38(14), 145021 (2021). https://doi.org/ 10.1088/1361-6382/abfed1

95. P. Drobiński, J. Lewandowski, Phys. Rev. D 96(12), 126011 (2017). https://doi.org/10.1103/ PhysRevD.96.126011

96. M. Varadarajan, Phys. Rev. D 61, 104001 (2000). https://doi.org/10.1103/PhysRevD.61.104001

97. M. Varadarajan, Phys. Rev. D 64, 104003 (2001). https://doi.org/10.1103/PhysRevD.64.104003

98. M. Varadarajan, Phys. Rev. D 66, 024017 (2002). https://doi.org/10.1103/PhysRevD.66.024017

99. A. Ashtekar, J. Lewandowski, H. Sahlmann, Class. Quant. Grav. 20, L11 (2003). https://doi.org/10.1088/0264-9381/20/1/103

100. M. Assanioussi, J. Lewandowski, Phys. Rev. D 105(10), 104025 (2022). https://doi.org/10.1103/PhysRevD.105.104025

101. A. Ashtekar, J. Lewandowski, Class. Quant. Grav. 18, L117 (2001). https://doi.org/10.1088/ 0264-9381/18/18/102

102. J. Lewandowski, A. Okolow, J. Math. Phys. 50, 123522 (2009). https://doi.org/10.1063/1.3265923

103. C.L. Duston, Class. Quant. Grav. 29, 205015 (2012). https://doi.org/10.1088/0264-9381/29/ 20/205015

104. C.L. Duston, in 13th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (2015), pp. 2177-2179. https://doi.org/10.1142/9789814

105. M. Villani, Class. Quant. Grav. 38(15), 155014 (2021). https://doi.org/10.1088/1361-6382/ ac0e1a

106. J. Aastrup, J.M. Grimstrup, R. Nest, Class. Quant. Grav. 26, 065011 (2009). https://doi.org/ 10.1088/0264-9381/26/6/065011

107. D. Oriti, Class. Quant. Grav. 33(8), 085005 (2016). https://doi.org/10.1088/0264-9381/33/8/ 085005

108. J.C. Baez, S. Sawin, J. funct. Anal. 150(1), 1-26 (1997)

109. J. Lewandowski, T. Thiemann, Class. Quant. Grav. 16, 2299 (1999). https://doi.org/10.1088/ 0264-9381/16/7/311

110. C. Fleischhack, Commun. Math. Phys. 249, 331 (2004). https://doi.org/10.1007/s00220-004- 1052-4

111. W. Fairbairn, C. Rovelli, J. Math. Phys. 45, 2802 (2004). https://doi.org/10.1063/1.1763247

112. C. Rovelli, M. Smerlak, Class. Quant. Grav. 29, 055004 (2012). https://doi.org/10.1088/0264-9381/29/5/055004

113. H. Minkowski, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1897, 198 (1897). http://eudml.org/doc/58391

114. K.M. Moerman, J. Open Source Softw. 3(22), 506 (2018). https://doi.org/10.21105/joss.00506

115. J. Marsden, T. Ratiu, Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems. Texts in Applied Mathematics (Springer, New York, 2002). https://books.google.com/books?id=I2gH9ZIs-3AC

116. A. Weinstein, J. Differ. Geom. 18(3), 523 (1983). https://doi.org/10.4310/jdg/1214437787

117. V. Aquilanti, H.M. Haggard, R.G. Littlejohn, L. Yu, J. Phys. A 40, 5637 (2007). https://doi.org/10.1088/1751-8113/40/21/013

118. M. Kapovich, J.J. Millson, J. Differ. Geom. 44(3), 479 (1996). https://doi.org/10.4310/jdg/ 1214459218

119. F. Girelli, E.R. Livine, Class. Quant. Grav. 22, 3295 (2005). https://doi.org/10.1088/0264- 9381/22/16/011

120. L. Freidel, J. Ziprick, Class. Quant. Grav. 31(4), 045007 (2014). https://doi.org/10.1088/ 0264-9381/31/4/045007

121. M. Dupuis, F. Girelli, E.R. Livine, Gen. Rel. Grav. 46(11), 1802 (2014). https://doi.org/10.1007/s10714-014-1802-3

122. L. Freidel, M. Geiller, J. Ziprick, Class. Quant. Grav. 30(8), 085013 (2013)

123. A.S. Cattaneo, A. Perez, Class. Quant. Grav. 34(10), 107001 (2017)

124. C. Rovelli, S. Speziale, Phys. Rev. D 82, 044018 (2010). https://doi.org/10.1103/PhysRevD.82.044018

125. J.V. José, E.J. Saletan, Classical Dynamics. A Contemporary Approach (Cambridge University Press, New York, NY, USA, 1998)

126. J.W. Barrett, L. Crane, J. Math. Phys. 39(6), 3296 (1998)

127. J.W. Barrett, L. Crane, Class. Quant. Grav. 17(16), 3101 (2000)

128. A. Barbieri, Nucl. Phys. B 518(3), 714 (1998). https://doi.org/10.1016/S0550- 3213(98)00093-5. https://www.sciencedirect.com/science/article/pii/S0550321398000935

129. J.C. Baez, J.W. Barrett, Adv. Theor. Math. Phys. 3(4), 815 (1999). https://doi.org/10.4310/ ATMP.1999.v3.n4.a3

130. H.M. Haggard, Asymptotic analysis of spin networks with applications to quantum gravity. Ph.D. thesis, University of California, Berkeley (2011)

131. H.M. Haggard, ILQGS talk: ”Quantization of the Volume of the Simplest Grain of Space”. https://relativity.phys.lsu.edu

132. S. Antu, C. Doran, H.M. Haggard, In preparation

133. S. Gukov, Commun. Math. Phys. 255, 577 (2005). https://doi.org/10.1007/s00220-005- 1312-y

134. S. Gukov, P. Sulkowski, JHEP 02, 070 (2012). https://doi.org/10.1007/JHEP02(2012)070

135. T. Dimofte, Adv. Theor. Math. Phys. 17(3), 479 (2013). https://doi.org/10.4310/ATMP.2013.v17.n3.a1

136. P. Norbury, Proc. Symp. Pure Math. 93, 41 (2015)

137. V. Bouchard, N.K. Chidambaram, T. Dauphinee, Commun. Numer. Theor. Phys. 12, 253 (2018). https://doi.org/10.4310/CNTP.2018.v12.n2.a2

138. G. Basar, G.V. Dunne, M. Unsal, JHEP 05, 087 (2017). https://doi.org/10.1007/ JHEP05(2017)087

139. A. Voros, in Annales de l'IHP Physique théorique, vol. 39 (1983), pp. 211-338

140. S.C. Creagh, J. Phys. A: Math. Gen. 27(14), 4969 (1994)

141. H.M. Haggard, Phys. Rev. D 87, 044020 (2013). https://doi.org/10.1103/PhysRevD.87.044020

142. C.E. Coleman-Smith, B. Müller, Phys. Rev. D 87, 044047 (2013). https://doi.org/10.1103/ PhysRevD.87.044047

143. A.P. Yutsis, I.B. Levinson, V.V. Vanagas, The Theory of Angular Momentum (S Monson, Jerusalem, 1962)

144. L. Freidel, E.R. Livine, J. Math. Phys. 51, 082502 (2010). https://doi.org/10.1063/1.3473786

145. L. Freidel, E.R. Livine, J. Math. Phys. 52, 052502 (2011). https://doi.org/10.1063/1.3587121

146. E.F. Borja, L. Freidel, I. Garay, E.R. Livine, Class. Quant. Grav. 28, 055005 (2011). https:// doi.org/10.1088/0264-9381/28/5/055005

147. L. Freidel, A. Perez, Universe 4(10), 107 (2018). https://doi.org/10.3390/universe4100107

148. L. Freidel, M. Geiller, D. Pranzetti, JHEP 11, 026 (2020). https://doi.org/10.1007/ JHEP11(2020)026

149. L. Freidel, M. Geiller, D. Pranzetti, JHEP 11, 027 (2020). https://doi.org/10.1007/ JHEP11(2020)027

150. L. Freidel, M. Geiller, D. Pranzetti, JHEP 01, 100 (2021). https://doi.org/10.1007/ JHEP01(2021)100

151. F. Fauvet, D. Manchon, S. Marmi, D. Sauzin, Resurgence, Physics and Numbers. Publications of the Scuola Normale Superiore (Scuola Normale Superiore, 2017). https://books.google.com/books?id=z0I_DwAAQBAJ

152. G.V. Dunne, M. Ünsal, PoS LATTICE2015, 010 (2016). https://doi.org/10.22323/1.251.0010

153. J. Gu, M. Marino, SciPost Phys. 15, 035 (2023). https://doi.org/10.21468/SciPostPhys.15.1.035, https://scipost.org/10.21468/SciPostPhys.15.1.035

154. V. Bonzom, M. Dupuis, F. Girelli, E.R. Livine, Deformed phase space for 3d loop gravity and hyperbolic discrete geometries (2014)

155. C. Charles, E.R. Livine, Class. Quant. Grav. 32(13), 135003 (2015). https://doi.org/10.1088/ 0264-9381/32/13/135003

156. M. Dupuis, L. Freidel, F. Girelli, A. Osumanu, J. Rennert, On the origin of the quantum group symmetry in 3d quantum gravity (2020)

157. V. Bonzom, M. Dupuis, F. Girelli, Q. Pan, Phys. Rev. D. 107(2), 026014 (2023)

158. H.M. Haggard, M. Han, W. Kamiński, A. Riello, Nucl. Phys. B 900, 1 (2015). https://doi.org/ 10.1016/j.nuclphysb.2015.08.023

159. H.M. Haggard, M. Han, W. Kaminski, A. Riello, Adv. Theor. Math. Phys. 23(4), 1067 (2019). https://doi.org/10.4310/ATMP.2019.v23.n4.a3

160. H.M. Haggard, M. Han, W. Kamiński, A. Riello, Phys. Lett. B 752, 258 (2016). https://doi.org/10.1016/j.physletb.2015.

161. H.M. Haggard, M. Han, A. Riello, Annales Henri Poincare 17(8), 2001 (2016). https://doi.org/10.1007/s00023-015-0455-4

162. L. Smolin, J. Math. Phys. 36, 6417 (1995). https://doi.org/10.1063/1.531251

163. S. Major, L. Smolin, Nucl. Phys. B 473, 267 (1996). https://doi.org/10.1016/0550- 3213(96)00259-3

164. L. Smolin, Quantum gravity with a positive cosmological constant (2002)

165. E. Witten, AMS/IP Stud. Adv. Math. 50, 347 (2011)

166. T. Dimofte, Commun. Math. Phys. 339(2), 619 (2015). https://doi.org/10.1007/s00220-015- 2401-1

167. S. Gukov, M. Marino, P. Putrov, Resurgence in complex Chern-Simons theory (2016)

168. P. Kirk, E. Klassen, Commun. Math. Phys. 153(3), 521 (1993)

169. J. Engle, R. Pereira, C. Rovelli, Phys. Rev. Lett. 99, 161301 (2007). https://doi.org/10.1103/ PhysRevLett.99.161301

170. L. Freidel, K. Krasnov, Class. Quant. Grav. 25(12), 125018 (2008)

171. W. Kaminski, M. Kisielowski, J. Lewandowski, Class. Quant. Grav. 27, 095006 (2010). https://doi.org/10.1088/0264-9381/29/4/049502.[Erratum: Class. Quant. Grav. 29, 049502 (2012)]

172. A. Perez, Living Rev. Rel. 16, 3 (2013). https://doi.org/10.12942/lrr-2013-3

173. S.K. Asante, B. Dittrich, H.M. Haggard, Phys. Rev. Lett. 125(23), 231301 (2020). https://doi.org/10.1103/PhysRevLett.

174. S.K. Asante, B. Dittrich, H.M. Haggard, Class. Quant. Grav. 38(14), 145023 (2021). https:// doi.org/10.1088/1361-6382/ac011b

175. S.K. Asante, B. Dittrich, J. Padua-Arguelles, Class. Quant. Grav. 38(19), 195002 (2021). https://doi.org/10.1088/1361-6382/ac1b44